# MATHEMATICS FOR ECONOMISTS

## B.A.(Economics) – Second Year

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DDE, Pondicherry University

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**BA II Year** **Paper Code: BAEC 1924**

# MATHEMATICS FOR ECONOMISTS

## Unit - I - Preliminaries

Elements of logic and proof – converse and contra-positive, necessary and sufficient conditions – mathematical induction – sets and set operations – Introduction to functions -composite functions, inverse function – Introduction to numbers

## Unit - II - Matrices And Determinants

Addition, scalar multiplication, matrix multiplication – the transpose – singularity and invertibility -Determinants – definition, properties, minors and cofactors

## Unit - III - Introduction to Vectors

Two, three and n – dimensional row and column vectors – vector addition and scalar multiplication – length of a vector, scalar products, orthogonality –linear and convex combinations of vectors..

## Unit - IV - Elementary Calculus - Differentiation

The derivative of a function – differentiability and continuity – techniques of differentiation – sums, products and quotients of functions – composite functions and the chain rule – inverse functions – implicit differentiation, second and higher order derivatives – concavity and convexity of functions

## Unit - V - Elementary Calculus - Integration

Introduction to integration– integration by substitution – integration by parts – area under a curve – properties of definite integrals

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# UNIT - I

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### 1.1.1 Objectives

- Interpret and explain the converse and contra-positive statements in mathematical contexts.
- Analyze and determine necessary and sufficient conditions in mathematical arguments.
- Explain the principles and steps involved in mathematical induction.
- Define sets and understand set operations.
- Describe the concept of functions and their role in mathematics.
- Explain composite functions and their properties.
- Define inverse functions and their relationship to original functions.
- Define and differentiate various types of numbers.

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### 1.1.2 Introduction

Logic is the discipline that deals with the methods of reasoning. One of the aims of logic is to provide rules by which we can determine whether particular reasoning or argument is valid. Logical reasoning is used in many disciplines to establish valid results. Rule of logic are used to provide proofs of theorems in mathematics, to verify the correctness of computer programs and to draw conclusions from scientific experiments.

### 1.1.3 Propositions

A declarative sentence (or assertion) which is either true or false but not both, is called a proposition (or statement). Sentences which are exclamatory, interrogative or imperative in nature are not propositions. Lower case letters such as p, q, r . . . are used to denote propositions. For example, we consider the following sentences:

1. Chennai is the capital of Tamil Nādu.
2. How beautiful is Rose?
3. 2+2=4
4. What time is it?
5. x+y=z

In the given statements, (2) and (4) are obviously not propositions as they are not declarative in nature. (l) and (3) are propositions, but (5) is not, since (1) is true,(3) is true and (5) is neither true nor false as the values of x, y, and z are not assigned.

If a proposition is true, we say that the truth value of that proposition is true, denoted by T or 1. If a proposition is false, the truth value is said to be false, denoted by F or 0.

**Syntax**

In Propositional logic there are two types of sentences- Simple sentences and compound sentences. Simple sentences express simple facts about the world. Compound sentences express logical relationships between the simple sentences of which they are composed.

Note that the constituent sentences within any compound sentence can be either simple sentences or compound sentences or a mixture of the two.

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**Definition: Atomic Statement**

Propositions which do not contain any of the logical operators or connectives are called atomic (primary or primitive) propositions. The area of logic that deals with propositions is called propositional logic or propositional calculus.

Simple sentences in Propositional Logic are often called proposition constants or, sometimes, logical constants. We write proposition constants as strings of letter, digits, and underscores ("_"), where the first character is a lower-case letter.

**For example:**

Raining is a proposition constant, as are rAiNiNg, r32aining, and raining_or_snowing. Raining is not a proposition constant because it begins with an upper Case character.324567 fails because it begins with a number. Raining-or-snowing fails because it contains hyphens (instead of underscores).

**Definition: Molecular Statement**

Mathematical statements which can be constructed by combining one or more atomic statements using connectives are called molecular or compound propositions. There are five types of compound sentences, viz. negations, conjunctions, disjunctions, implications, and biconditionals.

The following table gives a hierarchy of precedence for our operators. The ¬ operator has higher precedence than ; has higher precedence than V; V has higher precedence than

⇒; and ⇒ has higher precedence than ⇔.

¬ - Negations

∧ - Conjunctions

∨ - Disjunctions

⇒ - Implications

⇔ Biconditionals

### 1.1.4 Connectives

**Definition-Conjunction**

When p and q are any two propositions, the proposition conjunction of p and q is defined as the compound proposition that is true when both p and q are true and is false otherwise.

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The side table is the truth table for the conjunction of two propositions p and q viz., "p and q".

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Definition-Disjunction**

When p and q are any two propositions, the propositions "p or q" *denoted by* $p \vee q$ *and called the disjunction* of p and q is defined as the compound proposition that is false when both p and q are false and are true otherwise. The side table is the truth table for the disjunction of two propositions p and q, viz $p \vee q$

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Definition-Negation**

Given any proposition p, another proposition formed by writing "It is not the case that" or "It is false that" before p or by inserting the word 'not' suitably in p is called the negation of p and denoted by ~p (read as 'not p'). ~p is also denoted ¬P. It p is true, then ~p is false and if p is false, then ~p is true. Above table is the truth table for the negation of p.
For example, if p is the statement "New Delhi is in India",
then ¬P is given by
¬P: Itis not the case that New Delhi is in India.

| p | ¬p |
|---|---|
| T | F |
| F | T |

**Conditional Statement: [If… ……… then]**

Let p and q be any two statements. Then the statement

$p \to q$ *is called a* conditional statement (read as if p then q).



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$p \to q$ has a truth value F if p has the truth value T and q has the truth value F. In all the remaining cases it has the truth value T.

| P | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Example:**

p: Ram is a Computer Science student q: Ram studies Data Structure

$p \to q$ : If Ram is a Computer Science student, then he will study Data Structure.

The different situations where the conditional statements are applied are listed below.

a) If p then q

b) p only if q

c) q whenever p

d) q is necessary for p

e) q follows from p

f) q when p

g) p is sufficient for q

h) p implies q

**Definition: Converse, Contrapositive & Inverse Statements**

If $p \to q$ is a conditional statement, then

a. $q \to p$ is called the converse of $p \to q$

b. $\neg q \to \neg p$ is called the contrapositive of $p \to q$

c. $\neg p \to \neg q$ is called the inverse of $p \to q$

**Example:** Write the contrapositive, the converse, and the inverse of the implication "The home team wins whenever it is raining".

**Solution**: Let p: It is raining and q: The home team wins

Conditional Statement $p \to q$ : If it is raining then the home team wins.

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Contra positive $\neg q \rightarrow \neg p$ : If the home team does not win then it is not raining.
Converse ( $q \rightarrow p$ ): If the home team wins then it is raining.
Inverse ( $\neg p \rightarrow \neg q$ ): If it is not raining then the home team does not win.

### 1.1.5 Biconditional Proposition

If p and q are two propositions, then the proposition p if and only if q, denoted by p ↔ q is called the biconditional statement and is defined by the following truth table.
**Note:** p↔q is True if both p and q have the same truth values. Otherwise, p↔q is False.

| p | q | p↔q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Example:**

p: You can take the flight q: You can buy a ticket
p↔q: You can take the flight if and only if you buy a ticket

**Semantics**

Artificial symbolic language is called formal language, in which symbols are used to form formulas and formulas serve to express propositions.

Semantics is concerned with the meaning of expressions when the symbols are interpreted in a certain way. Interpretation of the propositional language and to make the formulas express propositions is semantics. Formulas are composed of atoms (proposition symbols) and connectives. Atoms are intended to express simple propositions.

The connectives have their intended meanings: negation, conjunction, disjunction, implication, and equivalence expression, respectively,"not", "and", "or", "if then" and "Iff". Hence, if formulas A and B express propositions A and B respectively, then the following non-atomic formulas on the left express the corresponding compound propositions on the right:

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| | |
|---|---|
| $\neg A$ | Not A |
| $A \wedge B$ | A and B |
| $A \vee B$ | A or B |
| $A \rightarrow B$ | If A then B |
| $A \leftrightarrow B$ | A Iff B |

**Syntax,** on the other hand, is concerned with the formal structure of expressions, irrespective of any interpretation.

**Symbolize the Statements using Logical Connectives Examples:**

1. The automated reply can be sent when the file system is full.
   p:The automated reply can be sent, q: The file system is full
   **Solution:** Symbolic form: $q \rightarrow p$
2. Write the symbolized form of the statement. If either Ram takes C ++or Kumar takes Pascal, then Latha will take Lotus.
   R: Ram takes C ++ K: Kumar takes Pascal L: Latha takes Lotus
   **Solution:** Symbolic form: $(R \vee K) \rightarrow L$
3. Let p, q, r represents the following propositions,
   p: It is raining, q: The sun is shining, r: There are clouds in the sky

**Symbolize the following statements.**

a. If it is raining, then there are clouds in the sky
b. If it is not raining, then the sun is not shining and there are clouds in the sky.
c. The sun is shining if and only if it is not raining.

**Solution:**

a) $p \rightarrow r$
b) $p \rightarrow (\neg q \wedge r)$
c) $q \leftrightarrow \neg r$

4. Symbolize the following statements:
   (i) If the moon is out and it is not snowing, then Ram goes out for a walk.
   (ii) If the moon is out, then if it is not snowing, Ram goes out for a walk.
   (iii) It is not the case that Ram goes out for a walk if and only if it is not snowing or the moon is out.

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**Solution:** Let the propositions be,

p: The moon is out.

q: It is snowing.

r: Ram goes out for a walk. Symbolic form:

(i) $(p \wedge \neg q) \rightarrow r$

(ii) $p \rightarrow (\neg q \rightarrow r)$

(iii) $\neg(r \leftrightarrow (\neg q \vee p))$

**Truth Assignment**

A truth assignment for Propositional Logic is a mapping that assigns a truth value to each of the proposition constants in the language. A truth assignment satisfies a sentence if and only if the sentences is true under that truth assignment according to rules defining the logical operators of the language.

**Evaluation**

Evaluation is the process of determining the truth values of a complex sentence, given a truth assignment for the truth values of proposition constants in that sentence.

### 1.1.6 Satisfiability or Satisfaction

Satisfaction is the process of determining whether or not a sentence has truth assignment that satisfies it.

**Problem**

Consider a truth assignment in which p is true, q is false, r is true. Use this truth assignment to evaluate the following sentences.

(a) $p \Rightarrow q \wedge r$

(b) $p \Rightarrow q \vee r$

(c) $p \wedge q \Rightarrow r$

(d) $p \wedge q \Rightarrow \neg r$

(e) $p \wedge q \Leftrightarrow q \wedge r$

**Solution**

(a) False

(b) True

(c) True
(d) True
(e) True

### 1.1.7 Construction of Truth Tables

**Problems:**

1. Show that the truth values of the formula $p \wedge (p \rightarrow q) \rightarrow q$ are independent of their components.

**Solution**: The truth table for the formula is

| p | Q | $p \rightarrow q$ | $p \wedge (p \rightarrow q)$ | $(p \wedge (p \rightarrow q)) \rightarrow q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

The Truth values of the given formula are all true for every possible Truth value of p and q. Therefore, the Truth value of the given formula is independent of their components.

2. Show that the Truth value of $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is independent of their components.

**Solution:**

| p | q | $P \rightarrow q$ | $\neg p \vee q$ | Ans |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | T | T |

The Truth values of the given formula are all true for every possible truth value of p and q. Therefore, the Truth value of the given formula is independent of their components.

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3. Construct a truth table for $(q \wedge ((p \rightarrow q)) \rightarrow p$

**Solution:**

| p | q | $P \rightarrow q$ | $q \wedge (p \rightarrow q)$ | $q \wedge (p \rightarrow q) \rightarrow p$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | T | F |
| F | F | T | F | T |

4. Construct a TRUTH table for $\neg (p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$

**Solution:**

| p | q | r | $q \wedge r$ | $p \vee (q \wedge r)$ | $(p \vee q)$ | $(p \vee r)$ | $(p \vee q) \wedge (p \vee r)$ | $\neg(p \vee (q \wedge r))$ | Ans |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | F | F |
| T | T | F | F | T | T | T | T | F | F |
| T | F | T | F | T | T | T | T | F | F |
| T | F | F | F | T | T | T | T | F | F |
| F | T | T | T | T | T | T | T | F | F |
| F | T | F | F | F | T | F | F | T | F |
| F | F | T | F | F | F | T | F | T | F |
| F | F | F | F | F | F | F | F | T | F |

**Exercise**

Construct the table for each of the following compound propositions:

1. $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$
2. $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
3. $((p \vee q) \wedge ((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$
4. $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

**Introduction to proofs:**

Proof: A proof is a valid argument that establishes the truth of a mathematical statement.

Theorem: A theorem is a statement that can be shown to be true

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1. **Direct Proofs:** A direct proof shows that a conditional statement $p \to q$ is true by showing that if $p$ is true then $q$ must also be true
2. **Proof by contraposition (or) Proof by contra positive(or) indirect proof:**

   In proof by contraposition of $p \to q$ we take $\neg q$ as a hypothesis and using axioms, definition together with rules of reference show that $\neg p$ must follow
3. **Vacuous Proof**

   To show that $p$ is false that proof is called vacuous proof of the conditional statement.
4. **Trivial Proof:** A proof of $p \to q$ that uses the fact $q$ is true is called a trivial proof.
5. **Proof by contradiction:**

   In proof by contradiction of $p \to q$ assume $\neg q \to \neg p$ to show that $\neg p$ is true
6. **Proofs of equivalence:**

   To prove a theorem that is biconditional statement, that is a statement of the form $p \leftrightarrow q$ we show that $p \to q$ and $q \to p$ are both true
7. **Counter examples:**

   A statement of the form $\forall x\ P(x)$ is false we need only find a counter example,that is an example x for which P(x) is false.

**Problems**

1. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction

   **Solution**: P : $\sqrt{2}$ is irrational

   To start a proof by contradiction (ie) $\neg p$ is true (ie) $\sqrt{2}$ is rational

   Now we have to show that $\neg p$ is true leads to a contradiction

   If $\sqrt{2}$ is rational there exist integers 'a' and 'b' the $\sqrt{2} = \frac{a}{b} \to (1)$

   where a and b do not have common factor

   Now $2 = \left(\frac{a}{b}\right)^2$ squaring (1)

   $2b^2 = a^2$ which gives $a^2$ is even

   $a^2$ is even implies a is even then $a = 2c$

   $2b^2 = (2c)^2 = 4c^2 \Longrightarrow b^2 = 2c^2$ that means $b^2$ is even



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Again using the fact that if the square of an integer is even then the integer must be even $\therefore$ b is even

$$2 = \frac{a}{b}$$

'a' and 'b' have common number '2' which gives the contradiction to (1)

$\therefore$ our assumption $\sqrt{2}$ is rational is wrong. Hence $\sqrt{2}$ is irrational

2. Prove that if n is a positive integer, then n is odd if and only if 5n + 6 is odd.

**Solution:** Case i: Assume n is odd.

Let n = 2k + 1 where k is a positive integer

$\therefore$ 5n + 6 = 5(2k + 1) + 6

= 10k + 11 = 2(5k + 5) + 1 which is an odd number.

Hence if n is odd then 5n + 6 is odd

Case ii: To prove the converse

Let n be even. i.e n = 2k where k is a positive integer.

Then 5n + 6 = 5(2k) + 6 = 2 (5k + 3) which is always even.

Thus 5n + 6 is odd if and only if n is odd.

3. Prove that square of an even number is an even number by (i) direct method (ii) indirect method and (iii) proof by contradiction

**Solution:**

Direct proof: $(p \rightarrow q)$ :

Let n be even i.e. n = 2k, where k is an integer.

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2) = \text{an even number.}$$

Indirect proof: $(\neg q \rightarrow \neg p)$

To prove that if n is odd then $n^2$ is odd

Let n be odd. i.e. n = 2k – 1

$$\therefore n^2 = (2k-1)^2 = 4k^2 - 4k + 1$$

$$= 2(2k^2 - 2k) + 1 = \text{odd number}$$

Hence if n is odd then $n^2$ is odd. Or if n is even then $n^2$ is even.

Proof by contradiction:



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Let q be F then if $\neg p \rightarrow \neg q$ is T implies $\neg p$ is F or p is T.

Assume $n^2$ to be even when n is odd. But if n is even we have proved that $n^2$ is even by the indirect method. Hence if $n^2$ is even then n is even and our assumption that $n^2$ is even when n is odd is wrong. So, if n is even then $n^2$ is even.

4. Prove that proposition P(0), where P(n) is the proposition "If n is a positive integer greater than 1, then $n^2 > n$". What kind of proof you use.

   **Solution:** Proposition P(0) is the implication "If 0 > 1, then $0^2 > 0$".

   Since the hypothesis 0 > 1 is false, the implication P(0) is automatically true. We have used Vacuous Proof to the above problem.

5. Let P(n) be "If a and b are positive integers with a ≥ b, then $a^n \geq b^n$. Show that the proposition P(0) is true

   **Solution:** Proposition P(0) is the implication "If a ≥ b, then $a^0 \geq b^0$

   Since $a^0 = b^0 = 1$, the conclusion of P(0) is true. Hence P(0) is true (using trivial proof).

**Exercise**

1. Prove that if n is a positive integer then n is odd if and only if 6n + 5 is odd.
2. Prove that square of an odd number is an odd number by

   (i) direct method (ii) indirect method and (iii) proof by contradiction.
3. Prove that $\sqrt{3}$ is irrational by giving a proof by contradiction.

### 1.1.8 Mathematical Induction:

One of the most basic methods of proof is Mathematical Induction, which is a method to establish the truth of a statement Induction, about all natural numbers. It will often help us to prove a general mathematical statement involving positive integers when a certain instance of that statement suggests a general pattern.

**Statement of the Principle of Mathematical Induction:**

Let P(n) be a statement or proposition involving the natural number 'n'.
We must go through two steps to prove that the statement P(n) is true for



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all natural numbers.

Step 1: We must prove that P (1) is true.

Step 2: By assuming P (k) is true, we must prove that P (k + 1) is also true.

**Note:**

The condition (i) is known as the **Basic step** and the condition (ii) is known as **Inductive step**

**Problems:**

1. Show that by mathematical induction 1+2+3+4+……n = $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   **Solution**: BASIS STEP: To prove P (1) is true.

$$\text{Since } \frac{1(1+1)}{2} = 1$$

   INDUCTIVE STEP: Assume that the result is true for P(K)

$$p(k) = \sum_{i\ 1}^{k} i = \frac{k(k\quad 1)}{2}$$

$$1+2+3+....+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$p(k+1) = \frac{(k+1)(k+2)}{2}$$

   Hence the result is derived using Mathematical induction method.

2. For all n ≥ 1, prove that $1^2+2^2+3^2+4^2+\ldots\ldots + n^2 = \frac{n(n+1)\ (2n+1)}{6}$

   **Solution**: Let the given statement be P(n)

   P(n): $1^2+2^2+3^2+4^2+\ldots\ldots + n^2 = \frac{n(n+1)\ (2n+1)}{6}$ For n=1,

   (i) To prove P (1) is true.

   P (1) = $\frac{1(1+1)\ (2\times 1+1)}{6}$ = 1 which is true.

   ∴ .P(n) is true. Where n = 1

   (ii) Assume that P(k) is true for some positive integer k, i.e.,

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$1^2+2^2+3^2+4^2+\ldots\ldots..+k^2=\dfrac{k(k+1)\ (2k+1)}{6}$ --------- (1)

We shall now prove that P(k+1) is also true. Now we have,

$(1^2+2^2+3^2+4^2+\ldots\ldots +k^2) + (k+1) = \dfrac{k(k+1)\ (2k+1)}{6} + (k+1)^2$
(Using (1))

$$= \frac{k(k+1)\ (2k+1)\ +6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

Thus P(k+1) is true, wherever P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all-natural number n.

3. Show that $n^3$ +2n is divisible by 3.

**Solution**: Let P(n): $n^3$+2n is divisible by 3

(i) To prove P (1) is true.

P (1): $1^3$ +2.1=3 is divisible by 3 is true. --------- (1)

(ii) Assume

P(k): $k^3$+2k is divisible by 3

Claim: P(k+1)

Now, P(k+1): $(k+1)^3+2(k+1)$

$=k^3 +3k^2 +3k+1+2k+2$

$=k^3 +3k^2 +3k+2k+3$

$=(k^3+2k) +3(k^2+k+1)$ -------- (2)

Since by Induction step (using (l)) $k^3$ + 2k is divisible by 3 and $3(k^2+k + l)$

is a multiple of 3, we have equation (2) is divisible by 3.

∴ P (k + l) is true.

By the principle of Mathematical induction, $n^3$ + 2n is divisible by 3

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4. Prove that $8^n - 3^n$ is a multiple of 5.

**Solution:** Let P(n): $8^n - 3^n$ is a multiple of 5.

(i) To prove :P (1) is true.P (1) = $8^1 - 3^1 = 5$ is a multiple of 5 which is true.

(ii) Assume P (k) = $8^k - 3^k$ is a multiple of 5 is true.

i.e., $8^k - 3^k = 5m$ where $m \in Z^+$

$=> 8^k = 5m+3^k$ --------- (1)

Claim: P (k+1) is true.

Now, P(k+1) = $8^{k+1} - 3^{k+1}$

$= 8^k 8 - 3^k 3$

$= (5m + 3^k).8 - 3^k.3$ (Using (1))

$= 5.8m + 8.3^k - 3.3^k$

$= 5.8m + 5.3^k = 5\,(8m + 3^k)$

Which is a multiple of 5 for all 'm'.

$\therefore$ P (k + 1) is true.

Hence, $8^n — 3^n$ is a multiple of 5 for all n.

5. Using mathematical induction prove that $(3^n+7^n-2)$ is divisible by 8, for $n \geq 1$

**Solution**: Let P(n): $(3^n+7^n-2)$ is a multiple of 8.

(1) To prove P (1) is true.

P (1) = $(3^1+7^1-2) \equiv 8$ which is divisible by 8 is true.

(2) Assume P (k) = $(3^k+7^k-2)$ is divisible by 8 is true. ---- (1)

Claim: P (k+1) is true.

Now, P (k+1) $= 3^{k+1} + 7^{k+1} - 2$

$= 3^k.3 + 7^k.7 - 2$

$= 3(3^k + 7^k - 2) + 4(7^k + 1)$ ----- (2)

Now, $7^k+1$ is an even number, for $k \geq 1$.

$\therefore 4\,(7^k + l)$ is divisible by 8.

Since 3 $(3^k+7^k-2)$ is divisible by 8 (Using (l)) and 4 $(7^k+ l)$ is divisible by 8, the RHS of (2) is divisible by 8.

$\therefore$ P (k + l) is true.

Hence, P(n): $(3^n+7^n-2)$ is a multiple of 8.

6. Show that $a^n-b^n$ is divisible by (a–b).

**Solution**: Let P(n): $a^n-b^n$ is divisible by (a–b).

(i) To prove P (1) is true

P (1) = $a^1-b^1$ is divisible by (a–b) is true.

(ii) Assume P (k) = $a^k-b^k$ is divisible by (a–b) is true. -------- (1)

Claim: P (k+1) is true.

Now, P (k+1) $= a^{k+1}-b^{k+1}$

$=a^k.a-b^k.b$

$=[m\ (a-b) +b^k]\ a-b^k.b$

$=am\ (a-b) +ab^k-bb^k$

$=(a-b)\ ma+ (a-b)\ b^k$

$=(a-b)\ [ma +b^k]$

Which is a multiple of (a–b).

$\therefore$ P (k+1) is true.

Hence, P(n): an-bn is divisible by (a–b).

7. Show that $2^n<n!$ for all $n\geq 4$.

**Solution:** Let P(n): $2^n<n!$

(i) To prove p (1) is true. Since $n\geq 4$.

P (4): $2^4< 4!$ is true

(ii) Assume P (k): $2^k<k!$ is true. -------- (1)

Claim: P (k+1) is true whenever P(k) is true.

From (1) $2^k<k!$

Multiplying both sides of (1) by 2,

We get $2.2^k< 2.k!$

i.e., $2^{k+1}< (k+1)\ k!$

$= (k+1)!$

$2^k+k<k+1!$

$\therefore$ P (k+1) is true when P (k) is true.



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Hence, P(n): $2^n < n!$ is true for all $n \geq 4$.

8. Using mathematical induction, Prove that $2+2^2+2^3+\ldots.+2^n=2^{n+1}-2$

**Solution**: Let P(n): $2+2^2+2^3+\ldots.+2^n$

(i) To prove p(1) is true

P (1): $2^1=2^{1+1}-2$ is true.

(ii) Assume P(k): $2+2^2+2^3+\ldots.+2^k=2^{k+1}-2$ is true. -------- (1)

Claim: P(k+1) is true.

P(k+1): $2+2^2+2^3+\ldots.+2^k+2^{k+1}$

$=2^{k+1}-2+2^{k+1}$ ....(Using(1))

=2. 2k+1−2

=2k+2−2

$\therefore$ P(k+1) is true.

Hence, $2+2^2+2^3+\ldots.+2^n=2^{n+1}-2$ is true for all n.

9. Show that $\frac{1}{1.2}+\frac{1}{2.3}+\ldots.+\frac{1}{n(n+1)}=\frac{n}{n+1}$ .

**Solution**: Let P(n): $\frac{1}{1.2}+\frac{1}{2.3}+\ldots.+\frac{1}{n(n+1)}$

(i) To prove p(1) is true

P (1): $\frac{1}{1.2}=\frac{1}{1.(1+1)}$ is true.

(ii) Assume P(k): $\frac{1}{1.2}+\frac{1}{2.3}+\ldots.+\frac{1}{k(k+1)}$

$=\frac{k}{(k+1)}$

Claim: P(k+1) is true.

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$$P(k+1)=\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

$\therefore$ P(k+1) is true.

Hence, $\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$ is true for all n.

10. For all n ≥ 1, prove that $1^2+2^2+3^2+4^2+.......+n^2=\frac{n(n+1)(2n+1)}{6}$

**Solution**: Let the given statement be P(n).

P(n): $1^2+2^2+3^2+4^2+......+n^2=\frac{n(n+1)(2n+1)}{6}$

(i) To prove p(1) is true For n=1,

$P(1)=\frac{1(1+1)(2\times1+1)}{6}=\frac{(1\times2\times3)}{6}=1$ which is true

therefore, P(n) is true. Where n = 1

(ii) Assume that P(k) is true for some positive integer k ,i.e.,

$1^2+2^2+3^2+4^2+........+k^2=\frac{k(k+1)(2k+1)}{6}$ ..............(1)

We shall now prove that P(k+1) is also true.

Now we have,

$(1^2+2^2+3^2+4^2+........+k^2)+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$ (Using (1))

$$=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$



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Thus P(k+1) is true, wherever P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural number n.

11. Show that 1+3+5+………. + (2n−1) =$n^2$

**Solution**:

(i) To prove P (1) is true

P (1) =1=$1^2$ is true

(ii) Assume that P(k) is true for some positive integer n=k

1+3+5+…..+(2k−1)=$k^2$ is true

Now, to prove for "k+1"

1+3+5+…..+(2k−1)+(2(k+1)−1)=$(k+1)^2$

We know that 1+3+5+… +(2k−1) = $K^2$ so,

1+3+5+…..+(2k−1)+(2(k+1)−1=$K^2$+(2(k+1)−1)

Expanding =$k^2$+2k+2−1

=$k^2$+2k+1

=$(k+1)^2$

They are same! So, it is true.

12. Prove the following by using the principle of mathematical induction for all n∈N 1.2.3+2.3.4+……..+n (n+1) (n+2) = $\frac{n(n+1)(n+2)(n+3)}{4}$

**Solution**: Let the given statement be P(n)

P(n)=1.2.3+2.3.4+………..+n(n+1)(n+2)= $\frac{n(n+1)(n+2)(n+3)}{4}$

(i) To prove P (1) is true

For n=1, P(1)=> 1.2.3= $\frac{1(1+1)(1+2)(1+3)}{4}$ ≡6 = 24/4=6 which is true.

Therefore, P(n) is true, where n=1

(i) Assume that P(k) is true for some positive integer k

P(k) =1.2.3+2.3.4+………+k(k+1) (k+2)

$$= \frac{k(k+1)(k+2)(k+3)}{4} \qquad \ldots\ldots\ldots\ldots.(1)$$

We shall now prove that P(k+1) is also true

P(k+1) =1.2.3+2.3.4+……….+ (k+1)(k+1+1)(k+1+2)

= 1.2.3+2.3.4+……….+ (k+1) (k+2) (k+3)     (Using (1))

= k (k+1) (k+2) (k+3) + 4

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Thus P(k+1) is true, whenever P(k) is true

Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

13. Show that if n ≥ 1, then 1.1!+2.2!+3.3!+…+n.n!= (n+1)!–1

**Solution**: Let P(n): 1.1!+2.2!+3.3!+…+n.n!=(n+1)!–1

(i) To prove p(1) is true

P(1): 1.1!=(1+1)!–1 is true.

(ii) Assume (k): 1.1!+2.2!+3.3!+…+k.k! = (k+1)!–1 is true.

Claim: P(k+1) is true .

To prove: 1.1!+2.2!+3.3!+…+k.k!+ (k+1)(k+1)!

= (k+1)! –1+ (k+1)(k+1)!     ------- (Using (1)

= (k+1)! [(1+k+1)]–1

= (k+1)! (k+2)–1= (k+2)! – 1

= ((k+1) +1)! - 1

∴ P(k+1) is true. By mathematical induction we have,

P(n): 1.1! +2.2! + 3.3! + … +n.n! = (n+1)!–1, n ≥ 1.

14. Use mathematical Induction, Prove that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{n}}>\sqrt{n}$ for $n \geq 2$

**Solution**: Let: P(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{n}}>\sqrt{n}$

(i) Assume P (2): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}=(1.707)>\sqrt{2}$ is true.

(ii) Assume P(k): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{k}}>\sqrt{k}$ is true.------(1)

Claim: P(k+1) is true.



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i.e., To prove $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$

Consider, $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$$=\sqrt{k}+\frac{1}{\sqrt{k+1}} \qquad \text{(Using(1))}$$

$$=\frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$$

$$=\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$

$$>\frac{\sqrt{k^2}+1}{\sqrt{k+1}}$$

$$>\frac{k+1}{\sqrt{k+1}}$$

$$=\sqrt{k+1}$$

i.e.., $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$

$\therefore$ P(k+1) is true.

By mathematical induction we have, P(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+.......+\frac{1}{\sqrt{n}}>\sqrt{n}$

15. Show that $3^{2n}+4^{n+1}$ is divisible by 5, for $n\geq 0$.

**Solution**: Let P(n): $3^{2n}+4^{n+1}$ is divisible by 5.

(i) To prove P(1) is true

Assume P (0): $3^0+4^1$ is divisible by 5 is true.

(ii) Assume P(k): $3^{2k}+4^{k+1}$ is divisible by 5 is true.

i.e, $3^{2k}+4^{k+1}=5m$ (m– is integer)

=> $3^{2k}=(5m-4^{k+1})$ --------- (1)

Claim: P(k+1) is true.

i.e, To prove : $3^{2(k+1)}+4^{(k+1)+1}$ is divisible by 5.

Consider, $3^{2(k+1)}+4^{(k+1)+1}=3^{2k}.3^2+4^{k+1}.4$

$= (5m-4^{k+1}).3^2+4^{k+1}.4$ (using 1)

$=5m.3^2-4^{k+1}.3^2+4^{k+1}.4$



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$=5m.9-5 \times 4^{k+1}$

$=5(9m-4^{k+1})$=Multiple of 5

$\therefore$ P (k + l) is true.

Therefore, by Mathematical Induction P(n): $3^{2n}+4^{n+1}$ is divisible by 5.

**Exercise:**

1. Using Mathematical induction, prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9, for $n \geq 1$.
2. Use mathematical Induction, Prove that $\sum_{m}^{n} 3^m = \frac{3^{n+1}-1}{2}$
3. For every positive integer n, prove that $7^n - 3^n$ is divisible by 4
4. Prove that $2^n > n$ for all positive integers n.
5. Using Mathematical induction, prove that $H_{2^n} \geq 1+\frac{n}{2}$ , where $H_k = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{k}$

### 1.1.9 Set and Set Operations

**Definition:**

A set is a well-defined collection of objects, The adjective 'well-defined' means that element is contained in the set under self-financing engineering colleges in a state, and science branch in a college are sets.

Capital letters A, B, C … are generally use $C_r$… to denote elements. If x is an element represented as $x \in A$. Similarly | $y \notin A$

**Notations:**

Usually a set is represented in two ways, namely, (1) roster notation and (2) set b notation.

In roster notation, all the elements of the set are listed, if possible, separated by commas and enclosed within braces.

A few examples of sets in roster notation are given as follows:

1 The set of $V$ all vowels in the English alphabet: $V = \{a,e,i,o,u\}$

2 The set of $E$ even positive integers less than or equal to 10 : $E = \{2,4,6,8,10\}$

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3 The set of positive integers less than

Note: The order in which a set's elements are listed is unimportant. Thus {1,2,3}, {2,1,3} and {3,2,1} represent the same set. In set builder notation, we define the elements of the set by specifying a property that they have in common.

A few examples of sets in set builder notation are given as follows:

1 The set $V = \{x \mid x$ is a vowel in the English alphabet$\}$ is the same as $V \equiv \{a,e,i,o,u\}$

2 The set $A = \{x \mid x = n2$ where n is a positive integer less than $6\}$ is the same as $A = \{1,4,9,16,25\}$

3 The set $B = \{x \mid x$ is an even positive integer not exceeding $10\}$ is

Note: The set V in example (1) is read as "The set of all $x$ such $t$

The following sets play an important role in discrete math

$N = \{0,1,2,3,\ldots\}$, the set of natural numbers

$Z = \{\ldots, -2,-1,0,1,2, \ldots\}$, the set of integers

$Z^+ = \{1,2,3, \ldots\}$, the set of positive integers

$Q \equiv \{ \frac{p}{q} \mid p \in z, q \in z, q \neq 0\}$, the set of rational numbers

$R$ = the set of real numbers

**Property:**

If a set S has n elements, then its power set has $2^n$ elements, viz. $|P(S)| = 2^n$

**Proof:**

Number of subsets of $S$ having no element, i.e., the null sets = 1 or $C(n, 0)$

Number of subsets of $S$ having 1 element = $C(n, 1)$

In general, the number of subsets of $S$ having $k$ elements = the number of ways choosing k elements from n elements = $C(n, k)$; $0 \le k \le n$.

Therefore,



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$|P(s)|$ = Total number of subsets of S

$$= C(n,0) + C(n,1) + C(n,2) + ..... + C(n,n) .........................................(1)$$

Now $(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b^2 + .. + C(n,n)b^n$ .......(2)

Putting a=b=1 in (2),we get

$$C(n,0) + C(n,1) + C(n,2) + ..... + C(n,n) = (1+1)^n = 2^n ..............................(3)$$

Using (3) in (1), we get $|P(s)| = 2^n$

**Cartesian product:**

If $A$ and $B$ are sets, the set of all ordered pairs whose first component belongs to second component belongs to $B$ is called the cartesian product of $A$ and $B$ and is denoted by $A \times B$. In other words, $A \times B = \{(a,b) / a \in A \,\&\, b \in B\}$

**SET OPERATIONS:**

(i) The union of two sets **A** and **B**, denoted by $A \cup B$, is the set of elements that belong to $A$ or to $B$ or to both, viz., $A \cup B = x \mid x \in A$ or $x \in B$.

(ii) The intersection of two sets **A** and **B**, denoted by $A \cap B$, is the set of elements that belong to both and . viz., $A \cap B = x \mid x \in A$ and $x \in B$

(iii) If $A$ and $B$ are any two sets, then the set of elements that belong to $A$ but do not belong to $B$ is called the difference of $A$ and $B$ or relative complement of $B$ with respect to $A$ and is denoted by A – B or A \ B. viz,, $A - B = x \mid x \in A$ and $x \notin B$

(iv) If $U$ is the universal set and $A$ is any set, then the set of elements which belong to $U$ but which do not belong to $A$ is called the complement of $A$ and is denoted by $A'$ or $A^c$ or viz., $A^c = x \mid x \in U$ and $x \notin A$

(v) If $A$ and $B$ are any two sets, the set of elements that belong to $A$ or $B$, but not to both is called the symmetric difference of $A$ and $B$ and is denoted by $A \oplus B$ or $A \Delta B$ or $A + B$. It is obvious that $A \oplus B = (A - B) \cup (B - A)$



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**Problems:**

1. Prove that ($\underline{A}$ – B) ∩ (C – B) = $\phi$

**Solution:**

$$(A-B)\cap(C-B)=\{x/x\in A \text{ and } x\notin C \text{ and } x\in C \text{ and } x\notin B\}$$
$$=\{x/x\in A \text{ and } (x\in \overline{C} \text{ and } x\in C) \text{ and } x\in \overline{B}\}$$
$$=\{x/(x\in A \text{ and } x\in\phi) \text{ and } x\in \overline{B}\}$$
$$=\{x/x\in\phi \text{ and } x\in \overline{B}\}$$
$$=\{x/x\in\phi\cap\overline{B}\}$$
$$=\{x/x\in\phi\}$$
$$=\phi$$

2. If **A, B** and **C** are sets, then analytically Prove that $A$ – ($B$ ∩ C) = ($A$ – $B$) ∪ ($A$ – $C$) Proof:

$$A-(B\cap C)=\{x/x\in A \,and\, x\notin(B\cap C)\}$$
$$=\{x/x\in A \,and\, (x\notin B\cup x\notin C\}$$
$$=\{x/x\in A \,and\, x\notin B\} \,or\, \{x/x\in A \,and\, x\notin C\}$$
$$=\{x/x\in(A-B) \,or\, x\in(A-C)\}$$
$$=(A-B)\cup(A-C)$$

3. If **A, B** and **C** are sets, then analytically Prove that $A$ ∩ ($B$ – C) = ($A$ ∩ $B$) – ($A$ ∩ C) Proof:

$$A\cap(B-C)=\{x\backslash x\in A \text{ and } x\in(B-C)\ \}$$
$$=\{x\backslash x\in A \text{ and } x\in B \,\&\, x\notin C\}$$
$$=\{x\backslash x\in A \text{ and } (x\in B \text{ and } x\notin C)\}$$
$$=\{x\backslash x\in(A\cap B\cap\overline{C}\}$$
$$=A\cap B\cap\overline{C}$$
$$(A\cap B)-(A\cap C)=\{x\backslash x\in(A\cap B) \text{ and } x\in\overline{(A\cap C)}\}$$
$$=\{x\backslash x\in(A\cap B) \text{ and } x\in\overline{A}\cup\overline{C}\}\ (\because \text{By De Morgan's Law})$$
$$=\{x\backslash x\in(A\cap B) \text{ and } x\in\overline{A} \text{ or } \overline{C})\}$$
$$=\{x\backslash[x\in(A\cap B) \text{ and } x\in\overline{A}] \text{ or } [x\in(A\cap B) \text{ and } x\in\overline{C}]$$
$$=\{x\backslash x\in(A\cap\overline{A}\cap B) \,or\, x\in(A\cap B\cap\overline{C})\}$$
$$=\{x\backslash x\in\varnothing \,or\, x\in(A\cap B\cap\overline{C})\}$$
$$=\{x\backslash x\in(A\cap B\cap\overline{C})\}$$
$$=A\cap B\cap\overline{C}$$

4. If A,B and C are sets, then analytically Prove that $A\times(B\cap C)=(A\times B)\cap(A\times C)$.

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Proof:

$$
\begin{aligned}
A\times(B\cap C) &= \{(x,y)\setminus x\in A \text{ and } y\in(B\cap C)\}\\
&= \{(x,y)\setminus x\in A \text{ and } (y\in B \text{ and } y\in C)\}\\
&= \{(x,y)\setminus (x\in A \text{ and } y\in B) \text{ and } (x\in A \text{ and } y\in C)\}\\
&= \{(x,y)\setminus (x,y)\in (A\times B)\cap(A\times C)\}\\
&= (A\times B)\cap(A\times C)
\end{aligned}
$$

5. If A,B,C and D are sets, then analytically Prove that $(A\cap B)\times(C\cap D)=(A\times C)\cap(B\times D)$.

Proof:

$$
\begin{aligned}
(A\cap B)\times(C\cap D) &= \{(x,y)\setminus x\in(A\cap B) \text{ and } y\in(C\cap D)\}\\
&= \{(x,y)\setminus (x\in A \text{ and } x\in B) \text{ and } (y\in C \text{ and } y\in D)\}\\
&= \{(x,y)\setminus (x\in A \text{ and } y\in C) \text{ and } (x\in B \text{ and } y\in D)\}\\
&= \{(x,y)\setminus (x,y)\in(A\times C) \text{ and } (x,y)\in(B\times D)\}\\
&= \{(x,y)\setminus (x,y)\in(A\times C)\cap(B\times D)\}\\
&= (A\times C)\cap(B\times D)
\end{aligned}
$$

**Exercise:**

1.If A,B andC are sets, then analytically Prove that $A-(B\cup C)=(A-B)\cap(A-C)$

2.If A and B are sets,then analytically Prove that $\overline{A\oplus B}=\overline{A}\oplus B=A\oplus\overline{B}$

3.If A,B and C are sets,then analytically Prove that $A\cap(B\oplus C)=(A\cap B)\oplus(A\cap C)$

4.If A,B and C are sets,then analytically Prove that $(A-B)-C=A-(B\cup C)$

5.If A,B and C are sets,then analytically Prove that $(A\cap C)\subseteq(B\cap C)$ and $(A\cap\overline{C})\subseteq(B\cap\overline{C})\rightarrow A\subseteq B$

## 1.1.10 Introduction to function

Functions play a fundamental role in set theory, a branch of mathematics that deals with the study of sets and their properties. In set theory, functions provide a way to establish relationships between sets.

**Definition of a Function:**

In set theory, a function is defined as a special type of relation between two sets, known as the domain and the codomain. A function assigns exactly one element from the codomain to each element in the domain. This means that for every input in the domain, there is a unique output in the codomain.

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**Notation:**

Functions in set theory are commonly represented using arrow notation. For example, if we have a function f that maps elements from set A to set B, we write it as: $f: A \rightarrow B$

**Elements of a Function:**

The elements involved in a function are:

a) Domain: The set of all possible input values for the function. It specifies the starting point of the function.

b) Codomain: The set that contains all possible output values for the function. It represents the set to which the function maps the elements.

c) Range: The set of all actual output values of the function. It is the subset of the codomain that contains the mapped elements.

d) Mapping: The association between the elements of the domain and the elements of the codomain. Each element in the domain is mapped to a unique element in the codomain.

**Injective, Surjective, and Bijective Functions:**

In set theory, functions can have different properties based on how they map elements. These properties include:

a) Injective (One-to-One): A function is injective if each element in the domain maps to a distinct element in the codomain. In other words, no two elements in the domain map to the same element in the codomain.

b) Surjective (Onto): A function is surjective if every element in the codomain has at least one corresponding element in the domain. In other words, the function covers the entire codomain.

c) Bijective: A function is bijective if it is both injective and surjective. It means that each element in the domain maps to a unique element in the codomain, and the function covers the entire codomain.

**Composite and Inverse Function:**

**Composition of Functions:**

In set theory, functions can be composed, meaning that the



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output of one function can serve as the input for another function. The composition of two functions, f and g, is denoted as g∘f, where g is applied to the output of f.

**Inverse Functions:**

An inverse function undoes the mapping of a given function. For a function f: A → B, an inverse function f^(−1): B → A exists if and only if f is bijective. The inverse function reverses the mapping, taking elements from the codomain back to the domain.

**Function Notation and Examples:**

Functions can be defined explicitly using set-builder notation or explicitly specifying the mapping for each element. Here is an example:

**a. Explicit Notation:**

f: A → B

f(x) = 2x, where A = {1, 2, 3} and B = {2, 4, 6}

The function f maps elements from A to B, doubling each element. For example, f (2) = 4.

**b. Set-Builder Notation:**

f: A → B

A = {x | x is an integer}, B = {x | x is an even integer}

f(x) = 2x, where x is an element in A.

This notation defines the function f as mapping all integers in A to even integers in B by doubling each element.

Understanding functions in set theory helps in analyzing and describing relationships between sets. Functions provide a mathematical framework to study mappings and transformations, making them a crucial concept in various branches of mathematics and computer science.

**Composite functions**

The important thing to remember when finding a composite function is the order in which the functions are written: fg ($x$) means first apply the function g to $x$, then apply the function to the result.

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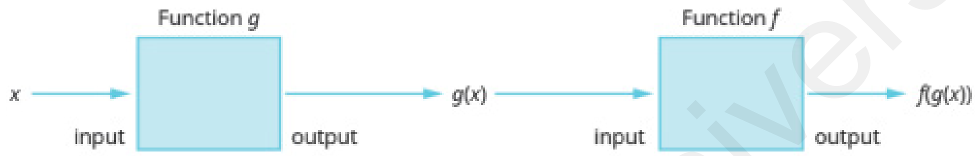
**Find and Evaluate Composite Functions**

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written f∘g and is defined by

(f∘g)(x)=f(g(x)).

We read f(g(x)) as " f of g of x. "




To do a composition, the output of the first function, g(x), becomes the input of the second function, f, and so we must be sure that it is part of the domain of f.

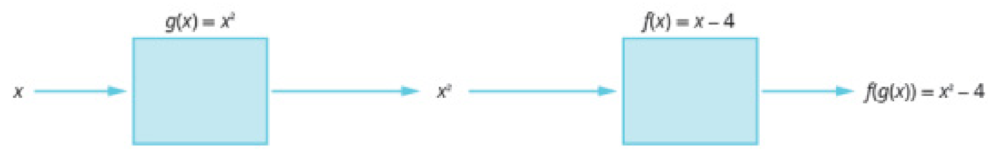
**Composition of Functions**

The composition of functions f and g is written f·g and is defined by

(f∘g)(x)=f(g(x))

We read f(g(x)) as f of g of x.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed g(x)=x^2 as a parabola and then shifted it down vertically four units, we were using the composition defined by (f∘g)(x)=f(g(x)) where f(x)=x−4.




**Example 1**

The functions *f*, *g*, and h are defined by:

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$$f(x) = x + 1$$
$$g(x) = x^2$$
$$h(x) = 3x$$

Find the following composite functions:

(i) fg($x$)

(ii) gh ($x$)

(iii) hgf ($x$)

(iv) f² ($x$)

**Solution**

(i) $fg(x) = f[g(x)]$
$= f(x^2)$
$= x^2 + 1$

Apply g followed by f; i.e. square, then add 1.

(ii) $gh\ (x) = g[h(x)]$
$= g(3x)$
$= (3x)^2$
$= 9x^2$

Apply h followed by g;i.e. multiply by 3 , then square.

(iii) $hgf\ (x) = hg[f(x)]$
$= h[g(x + 1)]$
$= h[(x + 1)^2]$
$= 3(x + 1)^2$

Apply followed by g followed by h; i.e. add 1, then square, then multiply by 3

(iv) $f^2(x) = f[f(x)]$
$= f(x + 1)$
$= (x + 1) + 1$
$= x + 2$

**Example 2**

The functions f and g are defined as:

$$f(x) = x^2 - 4 \quad x \geq 0$$
$$g(x) = \sqrt{x - 3} \quad x \geq 3$$



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(i) What is the range of each function?

(ii) Find the inverse function f¹, stating its domain.

(iii) Find the inverse function $g^{-1}$, stating its domain.

(iv) Write down the range of $f^{-1}$ and the range of $g^{-1}$.

**Solution**

(i) The range of $f$ is $f(x) \geq -4$. The range of $g$ is $g(x) \geq 0$.

(ii) The function can be written as

$$y = x^2 - 4$$
$$x = y^2 - 4$$
$$x + 4 = y^2$$
$$y = \sqrt{x+4}$$

Interchanging $x$ and $y$ : Rearranging : The domain of $f^{-1}$ is the same as the range of f.

$$f^{-1}(x) = \sqrt{x+4}\, x \geq -4$$

(iii) The function can be written as

$$y = \sqrt{x-3}$$
$$x = \sqrt{y-3}$$
$$x^2 = y - 3$$
$$y = x^2 + 3$$

Interchanging $x$ and $y$ The domain of $g^{-1}$ is the same as the range of g.

$$g^{-1}(x) = x^2 + 3\, x \geq 0$$

(iv) The range of $f^{-1}$ is $f^{-1}(x) \geq 0$ (the same as the domain of f)

The range of $g^{-1}$ is $g^{-1}(x) \geq 3$ (the same as the domain of g)

**Example 3:**

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$,

find: (a) $(f \circ \gamma)(x)$, $(b)(g \circ f)(x)$, and (c) $(f \cdot g)(x)$.

**Solution**

(a) Use the definition of $(f \circ g)(x)$. $(f \circ g)(x) = f(g(x))$



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$$(f \circ g)(x) = f(2x+3)$$

$$(f \circ g)(x) = 4(2x+3) - 5$$
$$(f \circ g)(x) = 8x + 12 - 5$$
$$(f \circ g)(x) = 8x + 7$$

(b) Use the definition of $(f \circ g)(x)$. $(g \circ f)(x) = g(f(x))$

$$(g \circ f)(x) = g(4x-5)$$
$$(g \circ f)(x) = 2(4x-5) + 3$$
$$(g \circ f)(x) = 8x - 10 + 3$$
$$(g \circ f)(x) = 8x - 7$$

(c) Notice that $(f - g)(x)$ is different than $(f \cdot g)(x)$. In part (a) we did the composition of the functions. Now in are not composing them, we are multiplying them.

Use the definition of $(f \,.\, g)(x)$.

Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$(f \cdot g)(x) = (4x-5) \cdot (2x+3)$$
$$(f \cdot g)(x) = 8x^2 + 2x - 15$$

**Example 4:**

For functions $f(x) = x^2 - 4$, and $g(x) = 3x + 2$, find: (a) $(f \circ g)(-3)$, (b) $(g \circ f)(-1)$, and (c) $(f \circ f)(2)$.

**Solution**

(a) Use the definition of $(f \circ g)(-3)$. $(f \circ g)(-3) = f(g(-3))$

$$(f \circ g)(-3) = f(3 \cdot (-3) + 2)$$
$$(f \circ g)(-3) = f(-7)$$
$$(f \circ g)(-3) = (-7)^4 - 4$$
$$(f \circ g)(-3) = 45$$

(b) Use the definition of $(g \circ f)(-1)$.

$$(g \circ f)(-1) = g(f(-1))$$
$$(g \circ f)(-1) = g((-1)^4 - 4)$$
$$(g \circ f)(-1) = g(-3)$$
$$(g \circ f)(-1) = 3(-3) + 2$$
$$(g \circ f)(-1) = -7$$

(c) Use the definition of $(f \circ f)(2)$.



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$$(f \circ f)(2) = f(f(2))$$
$$(f \circ f)(2) = f(2^2 - 4)$$
$$(f \circ f)(2) = f(0)$$
$$(f \circ f)(2) = 0^2 - 4$$
$$(f \circ f)(2) = -4$$

**Example 5:**

Verify that $f(x) = 5x - 1$ and $g(x)$ $\frac{x+1}{5}$ are inverse functions.

**Solution**

The functions are inverses of each other if $g(f(x) = x$ and $f(g(x)) = x$.

Find $g(5x-1)$ where $g(x) = \frac{x+1}{5}$, $\quad \frac{(5x-1)+1}{5} = x$

Simplify. $\quad \frac{5x}{5} = x$

Simplify. $\quad x = x$

$$f(g(x)) = x$$

Substitute $\frac{x+1}{5}$ for $g(x)$. $\quad f\left(\frac{x+1}{5}\right) = x$

Find $f\left(\frac{x+1}{5}\right)$ where $f(x) = 5x - 1$. $\quad 5\left(\frac{x+1}{5}\right) - 1$

$= \text{x}$

$(x+1) - 1 = \text{x} \quad \Rightarrow x = x$

**Example 6:**

Find inverse of $f(x) = 4x + 7$.

**Solution :**

| | | |
|---|---|---|
| Step 1. Substitute y for $f(x)$. | Replace $f(x)$ with $y$. | $f(x) = 4x + 7$<br>$y = 4x + 7$ |
| Step 2. Interchange the variables $x$ and $y$. | Replace $x$ with $y$ and then $y$ with $x$. | $x = 4y + 7$ |

| | | |
|---|---|---|
| Step 3. Solve for $y$. | Subtract 7 from each side. Divide by 4. | $x-7=4y$ <br> $\frac{x-7}{4}=y$ |
| Step 4. Substitute $f\ (x)$ for $y$. | Replace $y$ with $f''(x)$ | $\frac{x-7}{4}=f(x)$ |
| Step 5. Verify that the functions are inversers. | Show $f^{-}(f(x))=x$ and $f(f^{-}(x))=x$ | $f^{-1}(f(x)) = x$ <br> $\frac{(4x+7)-7}{4}(4x+7) = x$ <br> $\frac{4x}{4} = x$ <br> $x = x$ <br> $f(f^{-1}(x)) = x$ <br> $f\left(\frac{x-7}{4}\right) = x$ <br> $4\left(\frac{x-7}{4}\right)+7 = x$ <br> $x-7+7 = x$ <br> $x = x$ |

**Example 7:**

Find the inverse of $f(x)=\sqrt[5]{2x-3}$

**Solution:**

Substitute for $y$ for $f(x)$.

Interchange the variables $x$ and $y$.

Solve for $y$.

Substitute $f^{-1}(x)$ for $y$.

$$f(x) = \sqrt[5]{2x-3}$$
$$y = \sqrt[5]{2x-3}$$
$$x = \sqrt[5]{2y-3}$$
$$(x)^5 = (\sqrt[5]{2y-3})^5$$
$$x^5 = 2y-3$$
$$x^5+3 = 2y$$
$$\frac{x^5+3}{2} = y$$
$$f^{-1}(x) = \frac{x^5+3}{2}$$

Verify that the functions are inverses.



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$$
\begin{aligned}
f^{-1}(f(x)) &= x & f(f^{-1}(x)) &= x \\
f^{-1}(\sqrt[5]{2x-3}) &= x & f\left(\frac{x^5+3}{2}\right) &= x \\
\frac{(\sqrt[5]{2x-3})^5+3}{2} &= x & \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} &= x \\
\frac{2x-3+3}{2} &= x & \sqrt[5]{x^5+3-3} &= x \\
\frac{2x}{2} &= x & \sqrt[5]{x^5} &= x \\
x &= x & x &= x
\end{aligned}
$$

**Exercises:**

Find and evaluate Composite Functions and find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$, and (c) $(f \cdot g)(x)$.

1 $f(x) = 4x + 3$ and $g(x) = 2x + 5$

2 $f(x) = 6x - 5$ and $g(x) = 4x + 1$

3 $f(x) = 3x$ and $g(x) = 2x^2 - 3x$

4 $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

5 For functions $f(x) = 2x^2 + 3$ and $g(x) = 5x - 1$, find
   (a) $(f \circ g)(-2)$
   (b) $(g \circ f)(-3)$
   (c) $(f \circ f)(-1)$

### 1.1.11 Introduction to Numbers

Numbers are fundamental mathematical entities used for counting, measuring, and quantifying. The most basic type of number is the natural or counting numbers, which include 1, 2, 3, and so on. Whole numbers include the natural numbers along with zero (0), providing a complete set of non-negative numbers.

Integers encompass all whole numbers and their negative counterparts, including zero. Rational numbers are numbers that can be expressed as a fraction or ratio of two integers. They include fractions and terminating or repeating decimals.

Irrational numbers are numbers that cannot be expressed as fractions and have non-repeating decimal representations. Examples include $\sqrt{2}$ and $\pi$.

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Real numbers comprise both rational and irrational numbers and are used to represent quantities on a continuous number line.

Complex numbers involve a combination of real and imaginary numbers. They are written in the form a + bi, where a and b are real numbers and i is the imaginary unit ($\sqrt{-1}$).

Different number systems exist, such as binary (base–2), octal (base-8), and hexadecimal (base–16), which are used in computer science and digital systems.

Number properties, including commutative, associative, and distributive properties, govern the behavior of numbers in mathematical operations.

Prime numbers are integers greater than 1 that have no divisors other than 1 and themselves. They play a crucial role in number theory and cryptography.

Number sequences, such as arithmetic progressions (sequences with a constant difference between terms) and geometric progressions (sequences with a constant ratio between terms), have various applications in mathematics and real-world scenarios.

Number patterns, like the Fibonacci sequence and Pascal's triangle, exhibit recurring relationships and have intriguing mathematical properties.

| Word | Definition | Example |
|---|---|---|
| **Natural Numbers** | The numbers that we use when we are counting or ordering | {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 …} |
| **Whole Numbers** | The numbers that include natural numbers and zero. Not a fraction or decimal. | {0, 2, 3, 4, 5 6, 7, 8, 9, 10, 11 …} |

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| | | |
|---|---|---|
| **Integer** | A counting number, zero, or the negative of a counting number. No fractions or decimals | {… −3, −2, −1, 0, 1, 2, 3 …} |
| **Decimal Number** | Any number that contains a decimal point | 0.256 or 1.2 |
| **Rational Numbers** | Can be expressed as a fraction. Include integers and fractions or decimals | 1/2 , 2/3 , 4/7, 0.5, 6.7 |
| **Irrational Numbers** | Cannot be expressed as a fraction | Π, √2 … |
| **Positive** | Greater than 0. x is positive if $x > 0$. | 1, 17, 13.44, π, 18/3 |
| **Negative** | Less than 0. x is negative if $x < 0$. | −17, −18.892, −1981, −π |
| **Non-Negative** | Greater than **or equal to** 0. x is non-negative if $x \geq 0$. | 0, 1, π, 47812, 16/3, 189.53 |
| **Non-Positive** | Includes negative numbers and 0. | |
| **Even** | An integer that is divisible by 2. | 0; 2; −16; −8; 99837222 |
| **Odd** | An integer that is **NOT** divisible by 2. | 1; 7; 19; −17 |
| **Place Value** | It is the value of where the digit is in the number. Examples are units, tens, hundreds, thousands, ten thousands, hundred thousands, millions,… | |
| **Equivalent** | Equal (=) | ½ and 0.5 are equivalent |

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| | | |
|---|---|---|
| **Distinct** | Not equal. x and y are distinct if $x \neq y$. | 2 and 3 are distinct.<br><br>0 and 11 are distinct. $\pi$ and 3 are distinct. |
| **Constant** | A number that does not change | |
| **Consecutive (Evenly spaced)** | In a row; without any missing; numbers or objects are consecutive if none of them are skipped. | 1, 2, 3, and 4 are consecutive integers.<br><br>4, 6, 8, and 10 are consecutive even integers.<br><br>2008, 2009, and 2010 are consecutive years. |

### 1.1.12 Self Assessment question

1. Construct the table for each of the following compound propositions $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$.
2. Prove that $\sqrt{7}$ is irrational by giving a proof by contradiction.
3. Use mathematical Induction, Prove that $\sum_{m}^{n} 3^m = \dfrac{3^{n+1}-1}{2}$
4. Find and evaluate Composite Functions and find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$, and (c) $(f \cdot g)(x)$.
   $f(x) = 7x + 2$ and $g(x) = 2x + 54$.
5. If A and B are sets,then analytically Prove that $\overline{A \oplus B} = \overline{A} \oplus B = A \oplus \overline{B}$

### 1.1.13 Summary

**Connectives**

- **Conjunction**

| p | q | p ∧ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

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- **Disjunction**

| p | q | p∨q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

- **Negation**

| p | ¬p |
|---|---|
| T | F |
| F | T |

- **Conditional Statement: [If… ……… then]**

| P | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

- **Definition: Converse, Contrapositive & Inverse Statements**

If $p \to q$ is a conditional statement, then

a. $q \to p$ is called the converse of $p \to q$

b. $\neg q \to \neg p$ is called the contrapositive of $p \to q$

c. $\neg p \to \neg q$ is called the inverse of $p \to q$

- **Biconditional**

| p | q | P ↔ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Introduction to proofs:**

Proof: A proof is a valid argument that establishes the truth of a

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mathematical statement.

Theorem: A theorem is a statement that can be shown to be true

1. **Direct Proofs:** A direct proof shows that a conditional statement

   $p \rightarrow q$ is true by showing that if $p$ is true then $q$ must also be true

2. **Proof by contraposition (or) Proof by contra positive(or) indirect proof:**

   In proof by contraposition of $p \rightarrow q$ we take $\neg q$ as a hypothesis and axioms, definition together with rules of reference show that $\neg p$ must follow.

3. **Vacuous Proof**

   To show that $p$ is false that proof is called vacuous proof of the conditional statement.

4. **Trivial Proof:** A proof of $p \rightarrow q$ that uses the fact $q$ is true is called a trivial proof.

5. **Proof by contradiction:**

   In proof by contradiction of $p \rightarrow q$ assume $\neg q \rightarrow \neg p$ to show that $\neg\ p$ is true

6. **Proofs of equivalence:**

   To prove a theorem that is biconditional statement, that is a statement of the form $p \leftrightarrow q$ we show that $p \rightarrow q$ and $q \rightarrow p$ are both true

7. **Counter examples:**

   A statement of the form $\forall x\ P(x)$ is false we need only find a counter example,that is an example x for which P(x) is false.

- **Statement of the Principle of Mathematical Induction:**

  Let P(n) be a statement or proposition involving the natural number 'n'. We must go through two steps to prove that the statement P(n) is true for all natural numbers.

Step 1: We must prove that P (1) is true.

Step 2: By assuming P (k) is true, we must prove that P (k + l) is also true.

**Note:**

The condition (i) is known as the **Basic step** and the condition (ii) is



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known as **Inductive step**

- **SET OPERATIONS:**

(i) The union of two sets **A** and **B**, denoted by $A \cup B$, is the set of elements that belong to A or B to or to both,
viz., or $A \cup B = x \mid x \in A$ or $x \in B$.

(ii) The intersection of two sets **A** and **B**, denoted by $A \cap B$, is the set of elements that belong to both A and B. viz., $A \cap B = x \mid x \in A$ and $x \in B$

(iii) If $A$ and $B$ are any two sets, then the set of elements that belong to $A$ but do not belong to $B$ is called the difference of $A$ and $B$ or relation complement of $B$ with respect to $A$ and is denoted by
A – B or A\B. viz, $A - B = x \mid x \in A$ and $x \notin B$

(iv) If $U$ is the universal set and $A$ is any set, then the set of elements which belong to $U$ but which do not belong to $A$ is called the complement of $A$ and is denoted by $A'$ or $A^c$ or viz., $A^c = x \mid x \in A$ and $x \notin A$

(v) If $A$ and $B$ are any two sets, the set of elements that belong to $A$ or $B$, but not to both is called the symmetric difference of $A$ and $B$ and is denote by $A \oplus B$ or $A \Delta B$ or A + B. It is obvious that $A \oplus B = (A - B) \cup (B - A)$

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# UNIT – II

## Lesson 2.1 - Matrices and Determinants

### Structure



### 2.1.1 Objectives:

1) Understand the meaning and properties of addition, scalar multiplication, and matrix multiplication
2) Explain the concept of transpose and its properties
3) Apply addition, scalar multiplication, and matrix multiplication to solve problems
4) Use the transpose of a matrix to solve problems
5) Determine whether a matrix is singular or invertible
6) Determine determinants, minors, and cofactors of matrices

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### 2.1.2 Introduction:

Matrices and determinants are important mathematical concepts that are used in various fields such as engineering, physics, computer science, and economics.

A matrix is a rectangular array of numbers, arranged in rows and columns. Matrices represent data and perform mathematical operations such as addition, subtraction, multiplication, and inversion.

The addition of matrices involves adding corresponding entries of two matrices of the same size. For example, if A and B are two matrices of the same size, then A+B is the matrix whose $(i, j)^{th}$ entry is the sum of the $(i, j)^{th}$ entries of A and B.

Scalar multiplication of a matrix involves multiplying each entry of a matrix by a scalar. For example, if A is a matrix and k is a scalar, then kA is the matrix whose $(i, j)^{th}$ entry is k times the $(i, j)^{th}$ entry of A.

Matrix multiplication involves multiplying rows of the first matrix by columns of the second matrix to produce a new matrix. For example, if A and B are matrices such that the number of columns in A is equal to the number of rows in B, then AB is the matrix whose $(i, j)^{th}$ entry is the dot product of the $i^{th}$ row of A and the $j^{th}$ column of B.

The transpose of a matrix involves interchanging rows and columns. For example, if A is a matrix, then the transpose of A, denoted by $A^T$, is the matrix whose $(i, j)^{th}$ entry is the $(j, i)^{th}$ entry of A.

A matrix is singular if its determinant is zero, and invertible otherwise. Determinants are scalar values associated with square matrices that can be used to determine whether a matrix is invertible or singular.

Properties of determinants include linearity, alternate sign, and multiplication across rows or columns. The minor of an entry in a matrix is the determinant of the submatrix obtained by deleting the row and column containing that entry. The cofactor of an entry in a matrix is the minor multiplied by $(-1)^{(i+j)}$ where i and j are the row and column indices of the entry.

Minors and cofactors of a matrix are related to its determinants and can be used to find the inverse of a matrix. The inverse of matrix A is the



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matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix. The formula for the inverse of a matrix involves the adjugated of the matrix, which is the transpose of the matrix of cofactors of A.

### 2.1.3 Definition:

A system of $m \times n$ numbers arranged in the form of an ordered set of $m$ horizontal lines called rows & $n$ vertical lines called columns is called an $n$ matrix. The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots a_{1j} & \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots a_{2j} & \ldots a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{m \times n}$$

**Note:**

i) Matrices are generally denoted by capital letters.

ii) The elements are generally denoted by corresponding small letters.

### 2.1.4 Types of Matrices:

**(a) Rectangular matrix:**

Any $m \times n$ Matrix where $m \neq n$ is called rectangular matrix.

Ex:

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{2 \times 3}$$

**(b) Column Matrix:**

It is a matrix in which there is only one column.

Ex:

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3 \times 1}$$

**(c) Row Matrix:**

It is a matrix in which there is only one row.

Ex:



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$$\begin{bmatrix} 5 & 7 & 3 \end{bmatrix}_{3\times1}$$

**Square Matrix:** It is a matrix in which the number of rows equals the number of columns.

i.e its order is n x n.

Ex:

$$\begin{bmatrix} 4 & 5 \\ 7 & 2 \end{bmatrix}_{2\times2}$$

**(d) Diagonal Matrix:**

It is a square matrix in which all non-diagonal elements are zero.

Ex:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3}$$

**Scalar Matrix:** It is a square diagonal matrix in which all diagonal elements are equal.

Ex:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$$

**(e) Unit Matrix:** It is a scalar matrix with diagonal elements as unity and is denoted by I.

Ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

(f) **Null matrix (or) zero matrix:**

In a matrix, if all the elements are zero, then the matrix is called a null matrix or zero matrices and is denoted by O.

Ex:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

(g) **Upper Triangular Matrix:**

It is a **square** matrix in which all the elements below the principle diagonal are zero.

**Ex:**

$$\begin{bmatrix} 3 & 5 & 9 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$$

(**h**) **Lower Triangular Matrix:**

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It is a square matrix in which all the elements above the principle diagonal are zero.

Ex:

$$\begin{bmatrix} 3 & 0 & 0 \\ 4 & -6 & 0 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3}$$

## (i) Transpose of Matrix:

It is **a** matrix obtained by interchanging rows into columns or columns into rows.

Ex:

$$A = \begin{bmatrix} 2 & 6 & 8 \\ -1 & 4 & 5 \end{bmatrix}_{2\times 3} \qquad A^T = \begin{bmatrix} 2 & -1 \\ 6 & 4 \\ 8 & 5 \end{bmatrix}_{3\times 2}$$

## (j) Properties of Transpose:

1) $(A^T)^T = A$
2) $(A \pm B) = A^T \pm B^T$
3) $(kA)^T = kA^T$
4) $(AB)^T = B^T A^T$

**(j) Symmetric Matrix:**

If for a square matrix A, $A = A^T$ then A is symmetric.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 5 \\ 0 & 5 & 4 \end{bmatrix}_{3\times 3}$$

**Skew Symmetric Matrix :** If for a square matrix A, $A = -A^T$ then it is a skew-symmetric matrix.

$$A = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}_{3\times 3} \qquad A^T = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}_{3\times 3}$$

**Note:** For a skew Symmetric matrix, diagonal elements are zero.

**(k) Orthogonal matrix:**

If a square matrix satisfies the relation $AA^T = I$ then the matrix A is called an orthogonal matrix. & $A^T = A^{-1}$

The number of rows and columns in a matrix is called the order of the matrix. If a matrix A has m rows and n columns, then A is said to be of order m x n.

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### (l) Order of a Matrix

The number of rows and columns in a matrix is called the order of the matrix. If a matrix

A has m rows and n columns, then A is said to be of order m x n.

**Example**

$$1.\,A=\begin{bmatrix}7 & -9\\ 5 & 4\end{bmatrix}_{2\times 2} \qquad 2.\,B=\begin{bmatrix}1 & 4 & 3\\ 4 & 6 & 1\end{bmatrix}_{2\times 3} \qquad 3.\,C=\begin{bmatrix}3 & 8 & 9\\ 6 & 4 & 5\\ 7 & 8 & 1\end{bmatrix}_{3\times 3}$$

**(m) Equal matrices:**

Two matrices are said to be equal if and only if

(1) They have the same order

(2) Their corresponding entries are equal

### 2.1.5 Matrix Operations:

**Addition and Subtraction of matrices:**

Two matrices are considered conformable for addition when they have the same size. (Number of rows equal to the number of columns.

Thus if $A=\left[a_{i,j}\right]$ and $B=\left[b_{i,j}\right]$ are $m\times n$ then they can be added, and their sum is the matrix

$$A+B=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}+\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{bmatrix}$$

Similarly,

$$A-B=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}-\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}a_{11}-b_{11} & a_{12}-b_{12}\\ a_{21}-b_{21} & a_{22}-b_{22}\end{bmatrix}$$

### 2.1.6 Properties of Matrices under addition

The addition of matrices satisfies the following properties:

**(i) Commutative law**: If $A=\left[a_{i,j}\right]$ and $B=\left[b_{i,j}\right]$ are $m\times n$ then $A+B=B+A$

**(ii) Associative law**: For any three matrices

$A=\left[a_{i,j}\right]_{m\times n}$, $B=\left[b_{i,j}\right]_{m\times n}$ and $C=\left[c_{i,j}\right]_{m\times n}$, then $(A+B)+C=A+(B+C)$

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**(ii) Existence of additive identity:**

Let $A=\left[a_{i,j}\right]_{m\times n}$ be an $m\times n$ matrix and O be an $m\times n$ zero matrix, then $A+O=O+A$. In other words, O is the additive identity for matrix addition.

### 2.1.7 Additive Identity of a Matrix:

If A and B are two matrices of the same order and A+B=A=B+A, then matrix B is called the additive identity of a matrix. For any matrix A and zero matrices of the same order, then 0 is called the additive identity of A.

**Examples:**

$$\text{If } A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \text{ and O=}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$A+O=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}=A$$

$$O+A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}=A$$

### 2.1.8 Additive Inverse of a Matrix

If B are called additive inverse of each other, then A+B=O=B+A

Then A and and B are two matrices of same order.

Additive inverse of any matrix A is obtained by changing the sign of each non-zero entry of A.

## Example:

If $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$

Then $B=(-A)=\begin{bmatrix}-1 & -2\\ -4 & -3\end{bmatrix}$ is the additive inverse of $A$.

It can be verified as

$$A+B=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}+\begin{bmatrix}-1 & -2\\ -4 & -3\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

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### 2.1.9 Multiplication of Matrix by a Scalar

The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.

**Example:**

If $A = \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$ then $2A = \begin{bmatrix} 2(-1) & 2(3) \\ 2(2) & 2(-4) \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix}$

(a) **Properties of Scalar multiplication:**

$1. kA = Ak$

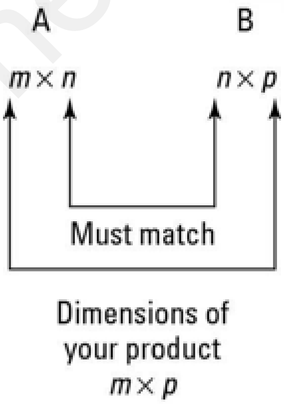
$2. k(A+B) = kA + kB$

$3. k(bA) = (kb)$ (where $k$ and $b$ are real numbers)

**(b) Multiplication of matrices:**

Two matrices A and B are conformable for multiplication, giving product AB if the number of columns of A is equal to the number of rows of B.

Thus if $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then the product $C = AB$ of the matrices $A$ and $B$ is an $m \times p$ matrix $C$ .




Thus,

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

## (c) Properties of Multiplication of Matrices:

**Example:**

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$$1. A(BC) = (AB)C$$
$$2. A(B \pm C) = AB \pm AC$$
$$3. (B+C)A = BA + CA$$
$$4. AB \neq BA$$

If $A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \end{bmatrix}$, then

$$C = AB = \begin{bmatrix} (3\times 2)+(1\times -1) & (3\times 1)+(1\times 0) & (3\times 3)\times(1\times 4) \\ (0\times 2)+(-2\times -1) & (0\times 1)+(-2\times 0) & (0\times 3)+(-2\times 4) \end{bmatrix}$$
$$C = \begin{bmatrix} 5 & 3 & 13 \\ 2 & 0 & -8 \end{bmatrix}$$

**Remark:**

If $AB$ is defined, then $BA$ *need* not be defined. In the above example, $AB$ *is* defined but $BA$ is not defined because $B$ has 3 columns while $A$ has only 2 ( and not 3) rows. If A, B are, respectively $m \times n$ and $k \times l$ matrices, then both $AB$ and $BA$ are defined **if and only if** $n = k$ and $l = m$. In particular, if both $A$ and $B$ are square matrices of the same order, then both $AB$ and $BA$ are defined.

In the above example both $AB$ and $BA$ *are* of different order and so $AB \neq BA$ . But one may think that perhaps $AB$ and $BA$ could be the same if they were of the same order. But it is not so, here we give an example to show that even if $AB$ and $BA$ are of same order, they may not be same.

## Example:

If $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Clearly $AB \neq BA$.

Thus, matrix multiplication is not commutative.

**Note:**

This does not mean that $AB \neq BA$ for every pair of matrices $A$, $B$ for which $AB$ and $BA$ , are defined. Observe that multiplication of diagonal matrices of same order will be commutative.

Problems:

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1. Simplify $\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}+\sin\theta\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}$.

**Solution:**

$$\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}+\sin\theta\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}=\begin{pmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\cos\theta\sin\theta\\ -\cos\theta\sin\theta+\cos\theta\sin\theta & \cos^2\theta+\sin^2\theta\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

2. If $X+Y=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}$ & $X-Y=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$ find $X$ & $Y$.

**Solutions.**

$$X+Y=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}------------(1)$$
$$X-Y=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}------------(2)$$

$$(1)+(2)\Rightarrow 2X=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}+\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$$
$$\Rightarrow 2X=\begin{pmatrix}10 & 0\\ 2 & 8\end{pmatrix}$$
$$\Rightarrow X=\begin{pmatrix}5 & 0\\ 1 & 4\end{pmatrix}$$
$$(1)-(2)\Rightarrow 2Y=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}-\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$$
$$\Rightarrow 2Y=\begin{pmatrix}4 & 0\\ 2 & 2\end{pmatrix}$$
$$\Rightarrow Y=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$$

3. Solve for $x, y, z, t$ if $2\begin{pmatrix}x & z\\ y & t\end{pmatrix}+3\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}=3\begin{pmatrix}3 & 5\\ 4 & 6\end{pmatrix}$.

**Solution:**



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$$2\begin{pmatrix} x & z \\ y & t \end{pmatrix} + 3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x & 2z \\ 2y & 2t \end{pmatrix} + \begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 12 & 18 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 12 & 18 \end{pmatrix}$$

$\Rightarrow 2x+3=9$
$\Rightarrow 2x=6$
$\Rightarrow x=3$

$\Rightarrow 2z-3=15$
$\Rightarrow 2z=18$
$\Rightarrow z=9$

$\Rightarrow 2y=12$
$\Rightarrow y=6$

$\Rightarrow 2t = 12$
$\Rightarrow 2t + 6 = 18$
$\Rightarrow t = 6$

4. If x $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$.Find the values of $x$ & $y$.

Solution:

$$x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\Rightarrow 2x - y = 10 \text{ \& } 3x + y = 5$$

Solving these equations, we get x=3,y=-4.

5. If $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ & $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ find k so that $A^2 = kA - 2I$.

Solution:

$$A^2 = kA - 2I$$

$$A^2 = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix}$$

$$A^2 = kA - 2I \Rightarrow \begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 3k & -2k \\ 4k & -2k \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix}$$

$\Rightarrow 3k - 2 = 1$
$\Rightarrow 3k = 3$
$\Rightarrow k = 1$

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6. If $A=\begin{pmatrix}2&0&1\\2&1&3\\1&-1&0\end{pmatrix}$ find $A^2-5A+6I$.

Solution:

$$A^2=\begin{pmatrix}2&0&1\\2&1&3\\1&-1&0\end{pmatrix}\begin{pmatrix}2&0&1\\2&1&3\\1&-1&0\end{pmatrix}=\begin{pmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{pmatrix}$$

$$A^2-5A+6I=\begin{pmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{pmatrix}-5\begin{pmatrix}2&0&1\\2&1&3\\1&-1&0\end{pmatrix}+6\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{pmatrix}-\begin{pmatrix}10&0&5\\10&5&15\\0&-1&-2\end{pmatrix}+\begin{pmatrix}6&0&0\\0&6&0\\0&0&6\end{pmatrix}$$

$$=\begin{pmatrix}1&-1&-3\\-1&-1&-10\\-5&4&4\end{pmatrix}$$

### 2.1.10 Transpose of a Matrix

The transpose of a matrix is a new matrix that is obtained by interchanging its rows and columns. In other words, if $A$ is an $m \times n$ matrix, then the transpose of $A$, denoted as $A^T$, is an $n \times m$ matrix.

Formally, the i,j-th entry of the transpose of $A$, denoted as $(A^T)_{ij}$, is equal to the j,i-th entry of $A$, i.e., $(A^T)_{ij} = A_{ji}$

Properties of transpose:

1. $(A^T)^T = A$: The transpose of the transpose of a matrix is the original matrix itself.

2. $(A+B)^T = A^T + B^T$: The transpose of a sum of matrices is equal to the sum of their transposes.

3. $(kA)^T = kA^T$: The transpose of a scalar multiple of a matrix is equal to the scalar multiple of the transpose of the matrix.

4. $(AB)^T = B^T A^T$ The transpose of a product of matrices is equal to the product of their transposes in reverse order.



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The diagonal elements of a square matrix remain the same after taking its transpose, i.e.,

$(A^T)_{ii} = A_{ii}$. The determinant of a matrix remains the same after taking its transpose, i.e.,

$$\det(A^T) = \det(A).$$

A matrix is symmetric if and only if it is equal to its transpose, i.e., A = $A^T$ .

The transpose operation has many important applications in linear algebra, such as solving systems of linear equations, computing the inverse of a matrix, and diagonalization of matrices.

## Examples 1:

Find the transpose of the matrix B. $B = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$.

## Solution:

To find the transpose of the $2 \times 2$ matix.

Let's switch the rows into columns and columns into rows. the resultant matrix is:

$$B^T = \begin{bmatrix} 5 & -2 \\ 6 & 3 \end{bmatrix}$$

## Example 2:

Find the transpose of the matrix A. $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \end{bmatrix}$.

## Solution:

To find the transpose of the given $2 \times 3$ matix. Lets write the rows as columns. The resultant matrix is of the order $3 \times 2$.

$$B = \begin{bmatrix} 2 & 0 \\ -1 & 5 \\ 3 & 2 \end{bmatrix}$$

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**Example 3:**

Verify whether

$$A = A^T \text{ if } A = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -7 \\ 1 & -7 & 9 \end{bmatrix}.$$

**Solution:**

Let us find the transpose of matrix Aby writing its rows as columns.

$$A^T = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -7 \\ 1 & -7 & 9 \end{bmatrix}$$

We can clearly see that $A = A^T$.

Note: We call A here a symmetric matrix.

**Exercise:**

Find the transpose of the given matrix B

$$\text{(i) } B = \begin{bmatrix} 1 & 5 \\ 3 & -4 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 7 & 8 & -3 \\ 5 & 9 & 1 \end{bmatrix} \quad \text{(iii) } \begin{bmatrix} -7 & 8 & 9 \\ 2 & -3 & -10 \\ 1 & 4 & 5 \end{bmatrix}$$

### 2.1.11 Singularity and Invertibility of a Matrix

## (a) Singularity of a Matrix:

A singular matrix is also called a degenerate matrix. A square matrix A is singular if it does not have an inverse, i.e., it is not invertible. In other words, if there exists no matrix B such that AB = BA = I, the matrix A is singular. Singularity is equivalent to having a determinant of zero (det(A) = 0).In a singular matrix, the rows or columns may be linearly dependent, leading to a loss of information.

**Invertibility of a matrix:**

An invertible matrix is also called a non-singular or non-degenerate matrix. A square matrix A is invertible if there exists another square matrix B such that AB = BA = I, where I is the identity matrix. The inverse of an invertible matrix A is denoted as $A^{-1}$.If a matrix A is invertible, it implies that its determinant is non-zero (det(A) ≠ 0).Inverse matrices are unique for a given matrix A.

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**(b) Properties and implications of Singularity and Invertibility of a Matrix:**

An invertible matrix can be thought of as a non-singular linear transformation, while a singular matrix represents a degenerate or non-invertible transformation. Invertible matrices have full rank, meaning that the rank of the matrix is equal to the number of rows or columns. A matrix is invertible if and only if its columns (or rows) are linearly independent.

If a matrix is singular, its null space (or kernel) is non-trivial, meaning that there exist non-zero vectors that are mapped to the zero vector by the matrix. Singular matrices can arise when there are redundant equations in a system or when the system is underdetermined or inconsistent.

These are the basic notes regarding the singularity and invertibility of matrices. They play a crucial role in various areas of mathematics, physics, engineering, and computer science, such as solving systems of linear equations, calculating determinants, and performing matrix operations.

### 2.1.12 Determinants

In this section, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and application of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equation of linear equations in two or three variables using inverse of a matrix.

## (a) Determinant of a matrix:

To every square matrix $a_{ij} = (i, j)^{th}$ we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A. If A be a square matrix then $A$ = determinant of A (i.e) $det\ A = |A|$.

If (i) $|A| = 0 \Rightarrow$ Matrix A is called Singular Matrix.

(ii) $|A| \neq 0, \Rightarrow$ Matrix A is called Non-Singular Matrix

**Note :** For non-singular matrix $A^{-1}$ exists.

**(b) Determinant of a matrix of order one:**

Let A=[a] be the matrix of order 1, then the determinant of A is defined to be equal to a.



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**(c) Determinant of a matrix of order two:**

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as:

$$\det A = |A| = \square = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

**Example:**

Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$.

**Solution:**

We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 8$.

**(d) Determinant of a matrix of order three:**

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows and three columns giving the same value as shown below.3

Consider the determinant of square matrix A =

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(i.e)

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Note:**

1. If A=kB where A and B are square matrices of order n, then $|A| = k^n \; |B|$, where $n = 1, \; 2, 3$.
2. For easier calculations, we shall expand the determinant along that row or column containing maximum zeros.



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**Example:1**

$$1 \text{ Evaluate } \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}.$$

**Solution:**

Note that in the third column, two entries are zero. So, expanding along third column (C3), we get

$$\square = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-1-12) - 0 + 0 = -52$$

**Example : 2**

$$\text{Evaluate } \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}.$$

**Solution:**

$$A = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= 0 - \sin\alpha(-\sin\beta\cos\alpha) - \cos\alpha(\sin\alpha\sin\beta) = 0$$

**Example: 3**

Find the values of $x$ for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

**Solution:** $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

We have $3 - x^2 = 3 - 8$

$$x^2 = 8$$

$$x = \pm 2\sqrt{2}$$

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**(e) Properties of Determinants:**

In the previous section, we have learnt how to expand the determinants. In this section, we will study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties are true for determinants of any order.

**Property:1** The value of the determinant remains unchanged if its rows and columns are interchanged.

Example: Verify the above property $\square = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

Solution: Expanding the determinant along the first row, we have

$$\square = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = -28$$

Property (1) verified.

**Property:2** If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

**Example:**

Verify the above property 2, $\square = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

**Solution:**

Expanding the determinant along the first row, we have

$$\Delta = 2\begin{vmatrix} 0 & 4 \\ 5 & 7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$$

By interchanging rows and columns, we get

$$\Delta = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix} = -28$$

Property (1) Verified.

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**Example:**

Verify the above property 2, $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

**Solution:**

Expanding the determinant along the first row, we have

$$\Delta = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = -28$$

By interchanging the rows (i.e) second and third row, we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = 28$$

$$\Delta = \Delta_1$$

**Property: 3** If any two rows (or columns) of a determinant are identical) (all corresponding elements are same), then value of determinant is zero.

**Example:**

Evaluate . $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

**Solution:**

Expanding the determinant along the first row, we have

$$\Delta = 3(6-6) - 2(6-9) + 3(4-6) = 0$$

**Property:4** If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

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Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then $\Delta_1 = k\Delta$

**Remark:**

If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \, (R_1 \text{and } R_2 \text{ are proportional})$$

**Example:**

Evaluate $\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

**Solution:**

$$\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

determinants.

**For example,**

$$\begin{vmatrix} a_{11}+k_1 & a_{12}+k_2 & a_{13}+k_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**Example:**

Evaluate $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$.



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**Solution:**

We have $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} =$

$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix} = 0+0=0$ (From Prop 3& 4)

1. For example,

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33a} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then $\Delta_1 = \Delta$

**Example:**

1. Evaluate the determinant of the following

$(i)\begin{pmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{pmatrix}$ $\qquad (ii)\begin{pmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{pmatrix}$

**Solution:**

(i) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 6\times7 & 1 & 6 \\ 4\times7 & 7 & 4 \\ 2\times7 & 3 & 2 \end{vmatrix} = 7\begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}$ [Taking out 7 common from $C_1$]

$= 7\times 0 = 0$ [$\because c_1$ and $c_3$ are identical]

$(ii)\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix} = \begin{vmatrix} -3\times(-2) & -3 & 2 \\ -1\times(-2) & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$ [Taking out -2 common from $c_1$]

$= (-2)\begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$

$= (-2)\times 0$ $\qquad$ [$\because c_1$ and $c_2$ are identical]

$= 0$



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2.Evaluate the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

**Solution:**

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \left[\text{Appling } c_3 \to c_2 + c_3\right]$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \left[\text{Taking } (a+b+c) \text{ common from } c_3\right]$$

$$= (a+b+c) \times 0 \ \left[\because c_1 \text{ and } c_3 \text{ are identical}\right]$$

$$= 0$$

3.Without expanding the determinant, Prove that $\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3$

**Solution:**

$$\text{L.H.S}=\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = \begin{vmatrix} 3x & 2x & x \\ 4x & 3x & 3x \\ 5x & 4x & 6x \end{vmatrix} + \begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix}$$

$$= x^3\begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y\begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ 6 & 4 & 6 \end{vmatrix}$$

$$= x^3\begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y \times 0 \ \left[\because c_1 \text{ and } c_2 \text{ are identical in II determinant}\right]$$

$$= x^3\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix} \ \left[\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2\right]$$

$$= x^3 \times (3-2) \quad [\text{Expanding along } c_1]$$

$$= x^3 = R.H.S$$

4. Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Solution:

$$\text{L.H.S}=\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix}$$

$[\text{Applying } c_1 \to c_1 + c_2 + c_3]$

$$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

[Taking (x+a+b+c) common from $c_1$]

$$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

$[\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

Expanding along $C_1$, we get

$$(x+a+b+c)\left[1(x^2)\right] = x^2(x+a+b+c) = \text{R.H.S}$$

2. Using properties of determinants, prove that $\begin{vmatrix} x+9 & x & x \\ x & x+9 & x \\ x & x & x+9 \end{vmatrix} = 243(x+3).$

**Solution:**

$$L.H.S = \begin{vmatrix} x+9 & x & x \\ x & x+9 & x \\ x & x & x+9 \end{vmatrix}$$

$$= \begin{vmatrix} 3x+9 & x & x \\ 3x+9 & x+9 & x \\ 3x+9 & x & x+9 \end{vmatrix} [\text{Applying } c_1 \to c_1+c_2+c_3]$$

$$= (3x+9)\begin{vmatrix} 1 & x & x \\ 1 & x+9 & x \\ 1 & x & x+9 \end{vmatrix}$$

$$= 3(x+3)\begin{vmatrix} 1 & x & x \\ 0 & 9 & 0 \\ 0 & -9 & 9 \end{vmatrix} [\text{Applying } R_2 \to R_2\text{-}R_1 \text{ and } R_3 \to R_3\text{-}R_2]$$

$= 3(x+3)\times 81$ [Expanding along $C_1$]

$= 24(x+3)$

$=$ R.H.S

**Exercise**:

Evaluate the determinant of the following

$$(i)\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} \quad (ii)\begin{vmatrix} c\text{os}\,\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \quad (iii)\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} \quad (iv)\begin{vmatrix} 5 & 2 & 0 \\ 1 & -3 & 2 \\ 4 & 0 & 1 \end{vmatrix}$$

If $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that $|3A| = 27|A|$.

Find the determinant of the following matrices

$$(i)\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix} (ii)\begin{bmatrix} x & 5-x \\ 4+x & x^2 \end{bmatrix} (iii)\begin{bmatrix} 1 & 7 & 3 \\ 4 & -5 & 6 \\ 2 & 1 & 0 \end{bmatrix} (iv)\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

4.Find the values of x,if

$$(i)\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \quad (ii)\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

Answers

$1.(i)-12 \quad (ii)1 \quad (iii)x^3-x^2+2 \; (iv)-1$

$3.(i)0 \quad (ii)x^3+x^2-x-2 \; (iii)120 \; (iv)-2$

$4.(i)\pm\sqrt{3} \; (ii)2$

### 2.1.13 Inverse of a Matrix

Minor of an element:

Consider a square matrix A of order n. Let $A = \left[a_{ij}\right]_{m\times n}$

The matrix is also can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & ....a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & ....a_{2j} & a_{2n} \\ .... & .... & .... & .... & ....... & ...... \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & .....a_{ij} & a_{in} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & ....a_{nj} & a_{mn} \end{bmatrix}$$



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**Example:**

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Consider

$M_{11}$ = Minor of an element $a_{11}$

$$(i.e)M_{11}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Similarly

$M_{12}$ = Minor of an element $a_{12}$

$$M_{12}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

**Example:**

$$A=\begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11}=\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix}, M_{12}=\begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix}, M_{13}=\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix}$$

$$M_{21}=\begin{vmatrix} 5 & 8 \\ 4 & 6 \end{vmatrix}, M_{22}=\begin{vmatrix} 2 & 8 \\ 0 & 6 \end{vmatrix}, M_{23}=\begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix}$$

**(b) Cofactor of an element:**

Let $A=\left[a_{ij}\right]_{m\times n}$

$C_{ij}=(-1)^{i+j}M_{ij}, M_{ij}$ is the minor of $a_{ij}$

$$If\ A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$C_{11}=\text{The cofactor of } a_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{12}=\text{The cofactor of } a_{12}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13}=\text{The cofactor of } a_{13}=(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and so on.

**Example:**

$$If A=\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

$$C_{11}=\text{The cofactor of } a_{11}=(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix}=5$$



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$$C_{12} = \text{The cofactor of } a_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix} = 3$$

$$C_{13} = \text{The cofactor of } a_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} = -6$$

Similarly, we get

**(c) Cofactor Matrix:**

A matrix $C = \left[C_{ij}\right]$ where $C_{ij}$ denotes cofactor of the element $a_{ij}$ of a matrix A of order nxn, is called a cofactor matrix.

In above matrix A, cofactor matrix is $C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$

**(d) Adjoint of Matrix:**

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A. Adjoint of A = (cofactor of matrix)$^T$

$$Adj = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Adjoint of a matrix A is denoted as AdjA. Thus if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{then Adj } A = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{then Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

### 2.1.14 Inverse of a Matrix (or) Reciprocal Matrix:

**(a) Invertibility**:

A matrix is invertible if it has an inverse. An inverse of a matrix A is denoted as $A^{-1}$ and satisfies the property that when multiplied with the original matrix A, it results in the identity matrix, denoted as I. An invertible matrix represents a transformation that preserves the properties of the original space, allowing for a unique solution to systems of equations. In practical terms, an invertible matrix can be used to solve linear equations and perform various mathematical operations.

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The invertibility of a matrix is closely related to its singularity. A matrix is invertible if and only if it is non-singular or non-degenerate, meaning its determinant is non-zero. If the determinant of a matrix is zero, it is singular, and it does not have an inverse. In other words, invertibility and singularity are opposite properties of matrices.

**(b) Inverse of the Matrix:**

If A is a non-singular matrix $\frac{1}{|A|} AdjA$ is defined to be the reciprocal of the matrix A or the Inverse of the matrix A. It is denoted by $A^{-1}$

$$A^{-1} = \frac{1}{|A|} Adj\ A.$$

Symbolically, it can be shown that $AA^{-1} = A^{-1}A = I$ .

**Example**

1. Find the minor and cofactor of 7 in the matrix $\begin{bmatrix} 3 & 7 & 4 \\ 2 & 6 & 3 \\ -3 & 5 & 1 \end{bmatrix}$.

**Solution:**

The give matrix is $A = \begin{bmatrix} 3 & 7 & 4 \\ 2 & 6 & 3 \\ -3 & 5 & 1 \end{bmatrix}$

We need to find the cofactor of 7

Cofactor of $7 = (-1)^{1+2} \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = (-1)^3 \left[2(1) - 3(-3)\right] = (-1)\left[2+9\right] = -11$

Therefore, the cofactor of 7 is -11.

**Example 2.**

Find the cofactor matrix for the matrix $\begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 5 & 1 \end{bmatrix}$.

**Solution:**

The given matrix is $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 5 & 1 \end{bmatrix}$

Cofactor of $3 = (-1)^{1+1} \begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} = (-1)^2 \left[(-1)\times 1 - 3\times 5\right] = (1)\left[-1-15\right] = -16$

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Cofactor of $0 = (-1)^{1+2}\begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = (-1)^3\left[(2)\times 1 - 3\times(-3)\right] = (-1)\left[2+9\right] = -11$

Cofactor of $4 = (-1)^{1+3}\begin{bmatrix} 2 & -1 \\ -3 & 5 \end{bmatrix} = (-1)^4\left[(2)\times 5 - (-1)\times(-3)\right] = (1)\left[10-3\right] = 7$

Cofactor of $2 = (-1)^{2+1}\begin{bmatrix} 0 & 4 \\ 5 & 1 \end{bmatrix} = (-1)^3\left[(0)\times 1 - (4)\times(5)\right] = (-1)\left[0-20\right] = 20$

Cofactor of $-1 = (-1)^{2+2}\begin{bmatrix} 3 & 4 \\ -3 & 1 \end{bmatrix} = (-1)^4\left[(3)\times 1 - (-3)\times(4)\right] = (1)\left[3+12\right] = 15$

Cofactor of

$3 = (-1)^{2+3}\begin{bmatrix} 3 & 0 \\ -3 & 5 \end{bmatrix} = (-1)^5\left[(3)\times 5 - (0)\times(-3)\right] = (-1)\left[15-0\right] = -15$

Cofactor of $-3 = (-1)^{3+1}\begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix} = (-1)^4\left[(0)\times 3 - (4)\times(-1)\right] = (1)\left[0+4\right] = 4$

Cofactor of $5 = (-1)^{3+2}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = (-1)^5\left[(3)\times 3 - (4)\times(2)\right] = (-1)\left[9-8\right] = -1$

Cofactor of $1 = (-1)^{3+3}\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = (-1)^6\left[(3)\times -1 - (0)\times(2)\right] = (1)\left[-3-0\right] = -3$

Therefore, the cofactor Matrix of given matrix

$$A = \begin{bmatrix} -16 & -11 & 7 \\ 20 & 15 & -15 \\ 4 & -1 & -3 \end{bmatrix}$$

**Example 3:**

Find the minor of the matix, such that the given matrix is $\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$.

**Solution:**

Let the given matrix be $A = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$.

Let us now find the minor of each of each element of this matix.

**Minor of 2=5**

(Element 2 is in the first row and the first column of the matrix. After excluding the first row and the first column we are left with element 5). row and the second column we are left with elements



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**Minor of 4=-3**

(Element 4 is in the first row and the second column of the matrix. After excluding the first row and the second column we are left with element -3).

**Minor of -3 =4**

(Element -3 is in the second row and the first column of the matrix. After excluding the second row and the first column we are left with element 4).

**Minor of 5 =2**

(Element 5 is in the second row and the second column of the matrix. After excluding the second row and the second column we are left with element 2).

Hence the Minor of Matrix $A = \begin{bmatrix} 5 & -3 \\ 4 & 2 \end{bmatrix}$.

**Example 4:**

Find the Co-factor matrix and the adjoint matrix for the given matrix

$$\begin{bmatrix} 5 & 9 & 2 \\ 1 & 8 & 5 \\ 3 & 6 & 4 \end{bmatrix}.$$

**Solution:**

The given matrix is $\begin{bmatrix} 5 & 9 & 2 \\ 1 & 8 & 5 \\ 3 & 6 & 4 \end{bmatrix}$.

Let us now first find the co-factors of each of the elements of the above matrix.

Co-factor of

$$9 = (-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = -(1(4) - 3(5)) = -(4 - 15) = 11.$$

Co-factor of

$$9 = (-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = -(1(4) - 3(5)) = -(4 - 15) = 11.$$

Co-factor of

$$2 = (-1)^{1+3} \begin{vmatrix} 1 & 8 \\ 3 & 6 \end{vmatrix} = -(1(6) - 3(8)) = 6 - 24 = -18.$$

Co-factor of

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$$1 = (-1)^{2+1}\begin{vmatrix} 9 & 2 \\ 6 & 4 \end{vmatrix} = -(9(4) - 2(6)) = -(36 - 12) = -24.$$

Co-factor of

$$8 = (-1)^{2+2}\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = +(5(4) - 3(2)) = 20 - 6 = 14.$$

Co-factor of

$$3 = (-1)^{3+1}\begin{vmatrix} 9 & 2 \\ 8 & 5 \end{vmatrix} = +(9(5) - 2(8)) = 45 - 16 = 29.$$

Co-factor of

$$3 = (-1)^{3+1}\begin{vmatrix} 9 & 2 \\ 8 & 5 \end{vmatrix} = +(9(5) - 2(8)) = 45 - 16 = 29.$$

Co-factor of

$$6 = (-1)^{3+2}\begin{vmatrix} 5 & 2 \\ 1 & 5 \end{vmatrix} = -(5(5) - 1(2)) = -(25 - 2) = -23.$$

Co-factor of

$$4 = (-1)^{3+3}\begin{vmatrix} 5 & 9 \\ 1 & 8 \end{vmatrix} = +(5(8) - 1(9)) = 40 - 9 = 31.$$

Co-factor Matrix= $\begin{bmatrix} 2 & 11 & -18 \\ -24 & 14 & -3 \\ 29 & -23 & 31 \end{bmatrix}$

Adjoint Matrix= $\begin{bmatrix} 2 & -24 & 29 \\ 11 & 14 & -23 \\ -18 & -3 & 31 \end{bmatrix}$.

**Example 5:**

Find the minor of the element 5 in the matrix $\begin{bmatrix} 2 & -3 & 7 \\ 4 & 5 & 1 \\ 6 & 0 & -4 \end{bmatrix}$.

**Solution:**

Let the given matrix be $A = \begin{bmatrix} 2 & -3 & 7 \\ 4 & 5 & 1 \\ 6 & 0 & -4 \end{bmatrix}$

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The aim is to find the minor of element 5. Element 5 lies in the second row and second column. Hence after excluding the elements of the second row and second column, we obtain the minor of element 5.

**Example 6:**

Find the co-factor matrix of the matrix $\begin{bmatrix} -4 & 7 \\ -11 & 9 \end{bmatrix}$.

**Solution:**

The given matrix $\begin{bmatrix} -4 & 7 \\ -11 & 9 \end{bmatrix}$ represents a $2 \times 2$ matrix.

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the co-factor matrix of A=. $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Hence the co-factor matrix of the given matrix is = $\begin{bmatrix} 9 & 11 \\ -7 & -4 \end{bmatrix}$.

### 2.1.15 Properties of Inverse of a Matrix

If A is non-singular matrix, then $(A^{-1})^{-1} = A$ is non-singular and $(A^{-1})^{-1} = A$.
If A is non-singular matrix, then $A^T$ is non-singular and $(A^{-1})^T = (A^T)^{-1}$.

If A and B both are non-singular matrices, then AAB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

If A is non-singular matrix, then $A^{-1}$ is non-singular and det. $(A^{-1}) = \dfrac{1}{\det(A)}$

**Example 1:**

Find $P^{-1}$,if it exists, where $P = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$.

**Solution:**

$$P^{-1} = \frac{adjP}{|P|}$$

$$|P| = \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} = 8 - 7 = 1$$

$$\text{Now, } adj(P) = (cofactor\ P)^T$$

$$(cofactor\ P)^T = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{adjP}{|P|} = \frac{\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}}{1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

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**Example 2:**

Find $A^{-1}$,if it exists, where $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

**Solution:**

$$A^{-1} = \frac{adjA}{|A|}$$

A Now, $|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$

Since, $|A| = 0, A^{-1}$ does not exist.

**Example 3:**

Find the inverse of A=$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$.

**Solution:**

The cofactors of entry $a_{11}$ is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = -4$

The minor of entry $a_{12}$ is $M_{12} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix} = \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix} = -80$

The cofactors of entry $a_{12}$ is $C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-80)$ .

$$C_{13} = -384$$
$$C_{21} = 7$$
$$C_{22} = -119$$
$$C_{23} = 420$$
$$C_{31} = -3$$
$$C_{32} = 39$$
$$C_{33} = 120$$

Hence, the matrix of cofactors of $[A]$ is $[C] = \begin{bmatrix} -4 & 80 & -384 \\ 7 & -119 & 420 \\ -3 & 39 & -120 \end{bmatrix}$

The adjoint of matrix $[A]$ is $[C]^T$ , $adj(A) = [C]^T = \begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$

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Hence,

$$[A]^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{-84}\begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.952 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

**Exercise:**

1. What is the minor of 5 in the matrix $\begin{bmatrix} 4 & 3 & 0 \\ 1 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$?

(a) 2 (b) 3 (c) 4 (d) 6

2. What is the cofactor of 2 in the matrix $\begin{bmatrix} 4 & 3 & 0 \\ 1 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$?

(a) 4 (b) 1 (c) 2 (d) -1

3.What is the minor of the element -2 in the matrix $\begin{bmatrix} 3 & 4 \\ 7 & -2 \end{bmatrix}$

(a) -2 (b) 7 (c) 4 (d) 3

4.What is the minor of the element 5 in the matrix $\begin{bmatrix} 3 & -4 & 5 \\ 0 & 7 & -2 \\ 6 & 4 & 1 \end{bmatrix}$?

(a) -42 (b) 42 (c) 38 (d) -46

5. What is the cofactor of 6 in the matrix $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 5 \\ 0 & 6 & 11 \end{bmatrix}$?

(a) 22 (b) -20 (c) 18 (d) -18

Answer 1. (c), 2. (b), 3. (d), 4. (a), 5. (d)

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**2.1.16 Self Assesment Questions**

1.If $A$ is a non-singular matrix such that $A^{-1}=\begin{bmatrix}5 & 3\\ -2 & -1\end{bmatrix}$, then find $\left(A^T\right)^{-1}$.

$$\left[\text{Ans:}(A^T)^{-1}=\begin{bmatrix}5 & -2\\ 3 & -1\end{bmatrix}\right]$$

2. Prove that $\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}=4a^2b^2c^2$

3.Determine the value of x+y if $\begin{bmatrix}2x+y & 4x\\ 5x-7 & 4x\end{bmatrix}=\begin{bmatrix}7 & 7y-13\\ y & x+6\end{bmatrix}$.

[Ans: x+y=5]

4. What is the cofactor of element 4 in the determinant $\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 5 & 8 & 9\end{vmatrix}$.

[Ans: Cofactor of element 4 is 6]

5. Find the inverse of $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & 2\\ -1 & 2 & 1\end{bmatrix}$.

$$\left[\text{Ans}: A^{-1}=\begin{bmatrix}-3/-16 & -4/-16 & 5/-16\\ -4/-16 & 0/-16 & -4/-16\\ 5/-16 & -4/-16 & -3/-16\end{bmatrix}\right]$$

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### 2.1.17 Summary

**1.Matrix:** A system of $m \times n$ numbers arranged in the form of an ordered set of $m$ horizontal lines called rows & $n$ vertical lines called columns is called an $m \times n$ matrix. The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2j} & \dots a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{m\times n}$$

**2.Types of matrix**

- **Rectangular matrix:** Any $m \times n$ Matrix where $m \neq n$ is called rectangular matrix
- **Column Matrix :**It is a matrix in which there is only one column.
- **Row Matrix:**It is a matrix in which there is only one row.
- **Square Matrix:** It is a matrix in which the number of rows equals the number of columns.
- **Diagonal Matrix:**It is a square matrix in which all non-diagonal elements are zero.
- **Scalar Matrix:** It is a square diagonal matrix in which all diagonal elements are equal.
- **Unit Matrix:** It is a scalar matrix with diagonal elements as unity and is denoted by I.
- **Null matrix (or) zero matrix:**In a matrix, if all the elements are zero, then the matrix is called a null matrix or zero matrices and is denoted by O.
- **Upper Triangular Matrix:**It is a square matrix in which all the elements below the principle diagonal are zero.
- **Lower Triangular Matrix:**It is a square matrix in which all the elements above the principle diagonal are zero.
- **Transpose of Matrix:**It is a matrix obtained by interchanging rows into columns or columns into rows.
- **Symmetric Matrix:** If for a square matrix A, $A = A^T$ then A is symmetric.

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- **Skew Symmetric Matrix :** If for a square matrix A, $A$ = - $A^T$ then it is a skew-symmetric

**Orthogonal matrix:**If a square matrix satisfies the relation $AA^T = I$ then the matrix A is called an orthogonal matrix. & $A^T = A^{-1}$

**3. Matrix Operations:**

- **Addition and Subtraction of matrices:**

Two matrices are considered conformable for addition when they have the same size. (Number of rows equal to the number of columns.

Thus if $A=\left[a_{i,j}\right]$ and $B=\left[b_{i,j}\right]$ are $m\times n$ then they can be added, and their

sum is the matrix $A+B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{bmatrix}$

Similarly,

$$A-B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}-\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22}\end{bmatrix}$$

- **Multiplication of matrix by a scalar:**

The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.

**Properties of Multiplication of Matrices:**

$$1. A(BC)=(AB)C$$
$$2. A(B\pm C)=AB\pm AC$$
$$3. \left(B+C\right)A=BA+CA$$
$$4. AB\neq BA$$

## 4.Transpose of a Matrix:

The transpose of a matrix is a new matrix that is obtained by interchanging its rows and columns. In other words, if $A$ is an $m \times n$ matrix, then the transpose of $A$, denoted as $A^T$, is an $m \times n$ matrix. Formally, the i, j-th entry of the transpose of $A$, denoted as $(A^T)_{ij}$, is equal to the j, i-th entry of $A$, i.e., $(A^T)_{ij} = A_{ji}$

**Properties of transpose:**

1. $(A^T)^T = A$ : The transpose of the transpose of a matrix is the original matrix itself.



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2. $(A+B)^T = A^T + B^T$ The transpose of a sum of matrices is equal to the sum of their transposes.
3. $(kA)^T = kA^T$ : The transpose of a scalar multiple of a matrix is equal to the scalar multiple of the transpose of the matrix.
4. $(AB)^T = B^T A^T$ The transpose of a product of matrices is equal to the product of their transposes in reverse order.

**5. Determinant of a matrix:**

To every square matrix $A = \left[a_{ij}\right]_{m\times n}$ we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i,j)^{th}$ element of A.

If A be a square matrix then $A$ = determinant of A (i.e) $det\ A = |A|$.

If $(i)\ |A| = 0 \Rightarrow$ Matrix A is called Singular Matrix.
(ii) $|A| \neq 0, \Rightarrow$Matrix A is called Non-Singular Matrix

**Properties of Determinants:**

**Property:1** The value of the determinant remains unchanged if its rows and columns are interchanged.

**Property:2** If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

**Property: 3** If any two rows (or columns) of a determinant are identical) (all corresponding elements are same), then value of determinant is zero.

**Property:4** If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

**Property:5** If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, the determinant can be expressed as sum of two (or more).

**Property:6** If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row(or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $Ri \rightarrow Ri + kRj\ \ (or)\ Ci \rightarrow Ci + kCj.$

## 6.Inverse of a matrix:

- **Minor of an element:**
- The matrix is also can be written as



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$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & ....a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & ....a_{2j} & a_{2n} \\ .... & .... & .... & .... & ....... & ...... \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & .....a_{ij} & a_{in} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & ....a_{nj} & a_{mn} \end{bmatrix}$$

**Example**:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$M_{11}$ = Minor of an element $a_{11}$

Consider $(i.e)M_{11}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Similarly

$M_{12}$ = Minor of an element $a_{12}$

$$M_{12}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

**Cofactor of an element:**

Let $A=\left[a_{ij}\right]_{m\times n}$

$C_{ij}=(-1)^{i+j}M_{ij}, M_{ij}$ is the minor of $a_{ij}$

$$If\ A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$C_{11}$ = The cofactor of $a_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$C_{12}$ = The cofactor of $a_{12}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$C_{13}$ = The cofactor of $a_{13}=(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

and so on.

**Adjoint of Matrix:**

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A. Adjoint of A = (cofactor of matrix) $^T$

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$$Adj = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Adjoint of a matrix A is denoted as AdjA. Thus if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{then Adj } A = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{then Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Inverse of the Matrix:**

If A is a non-singular matrix $\frac{1}{|A|} AdjA$ is defined to be the reciprocal of the matrix A or the Inverse of the matrix A. It is denoted by $A^{-1}$

$$A^{-1} = \frac{1}{|A|} Adj\ A.$$

Symbolically, it can be shown that $AA^{-1} = A^{-1}A = I$ .

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# UNIT – III

## Lesson 3.1- Introduction to Vectors

### Structure



### 3.1.1 Objectives:

- Define two, three, and n-dimensional row and column vectors.
- Interpret and explain vector addition and scalar multiplication in the context of vectors.
- Define the length of a vector and how to compute it.
- Define scalar products and their properties.
- Explain the properties and applications of scalar products and orthogonality
- Define linear combinations of vectors.
- Describe the properties and applications of convex combinations of vectors.

### 3.1.2 Introduction

Vectors and matrices are notational conveniences for dealing with systems of linear equations and in particular, they are useful for compactly representing and discussing the linear programming

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$$\text{Maximize} \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \ (i = 1,2, \dots, m),$$
$$x_j \geq 0 \ (j = 1,2, \dots, n).$$

This appendix reviews several properties of vectors and matrices that are, especially real problems. We should note, however, that the material contained here is more technical than understanding the rest of this book. It is included for completeness rather than for background.

### 3.1.3 Vectors

We begin by defining vectors, relations among vectors, and elementary vector operations.

Definition. A $k$-dimensional vector $y$ is an ordered collection of $k$ real numbers $y_1, y_2, \cdots$ written as $y = (y_1, y_2, \dots, y_k)$. The numbers $y_j \, (j = 1,2, \dots, k)$ are called the component vector $y$.

Each of the following are examples of vectors:

1. $(1, -3,0,5)$ is a four-dimensional vector. Its first component is 1 , its second component is third and fourth components are 0 and 5 , respectively.
2. The coefficients $c_1, c_2, \dots, c_n$ of the linear-programming objective function determine the $n$ vector $c = (c_1, c_2, \dots, c_n)$
3. The activity levels $x_1, x_2, \dots, x_n$ of a linear program define the $n$ -dimensional vector $x = (x_1, x_2, \dots, x_n)$.
4. The coefficients $a_{i1}, a_{i2}, \cdots, a_{in}$ of the decision variables in the $i$ th equation of a linear program an $n$-dimensional vector $A^i = (a_{i1}, a_{i2}, \dots, a_{in})$.
5. The coefficients $a_{1j}, a_{2j}, \dots, a_{nj}$ of the decision variable $x_j$ in constraints 1 through $m$ of a linear program define an $m$ -dimensional vector which we denote as $A_j = (a_{1j}, a_{2j}, \dots, a_{mj})$. Equality and ordering of vectors are defined by comparing the vectors' individual components. Formally, let $y = (y_1, y_2, \dots, y_k)$ and $z = (z_1, z_2, \dots, z_k)$ be two $k$-dimensional vectors.



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We write:

$$y = z \text{ when } y_j = z_j \ (j = 1,2,\ldots,k),$$
$$y \geq z \text{ or } z \leq y \text{ when } y_j \geq z_j \ (j = 1,2,\ldots,k),$$
$$y > z \text{ or } z < y \text{ when } y_j > z_j \ (j = 1,2,\ldots,k),$$

and say, respectively, that $y$ equals $z$, $y$ is greater than or equal to $z$ and that $y$ is greater than $z$. In the last two cases, we also say that $z$ is less than or equal to $y$ and less than $y$. It should be emphasized that not all vectors are ordered. For example, if $y = (3,1,-2)$ and $x = (1,1,1)$, then the first two components of $y$ are greater than or equal to the first two components of $x$ but the third component of $y$ is less than the corresponding component of $x$.

Note: 0 is used to denote the null vector $(0,0,\ldots,0)$, where the dimension of the vector is understood from context. Thus, if $x$ is a $k$-dimensional vector, $x \geq 0$ means that each component $x_j$ of the vector $x$ is nonnegative. We also define scalar multiplication and addition in terms of the components of the vectors.

**Definition:**

Scalar multiplication of a vector $y = (y_1, y_2, \ldots, y_k)$ and a scalar $\alpha$ is defined to be a new vector $z = (z_1, z_2, \ldots, z_k)$, written $z = \alpha y$ or $z = y\alpha$, whose components are given by $z_j = \alpha y_j$.

**Definition:**

Vector addition of two $k$-dimensional vectors $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ is defined as a new vector $z = (z_1, z_2, \ldots, z_k)$, denoted $z = x + y$, with components given by $z_j = x_j + y_j$.As an example of scalar multiplication, consider

$$4(3,0,-1,8) = (12,0,-4,32),$$

and for vector addition,

$$(3,4,1,-3) + (1,3,-2,5) = (4,7,-1,2).$$

Using both operations, we can make the following type of calculation:

$$\begin{aligned}(1,0)x_1 + (0,1)x_2 + (-3,-8)x_3 &= (x_1, 0) + (0, x_2) + (-3x_3, -8x_3)\\ &= (x_1 - 3x_3, x_2 - 8x_3).\end{aligned}$$

It is important to note that $y$ and $z$ must have the same dimensions for vector



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addition and vector comparisons. Thus $(6,2,-1)+(4,0)$ is not defined, and $(4,0,-1)=(4,0)$ makes no sense at all.

Let $R$ denote a set of real numbers. All real numbers are located at a straight line take position. By $R^2$ we denote the set of all ordered pairs of real numbers. $R^2$ may be regarc we can associate a point in the plane with each ordered pair. $R^2$ may also be taken as matrices. Thus, the elements of $R^2$ may just be considered vectors and thus $R^2$ may be defined vectors in the plane. Now it is obvious that if $X$ is a vector in a plane then it can express $\begin{bmatrix} x \\ y \end{bmatrix}$ or $(x,y)$ or $xi+yj$ so we can use the three representations of $X$ interchangeably. Similarly, all the vectors in space. Thus if $X$ is any vector in space, then it also can be written in three different ways as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $(x,y,z)$ or $xi+yj+zk$. In general, $R^n$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ - \\ x_n \end{bmatrix}$

Sometimes, we may call $R^2$ as the set of all 2-vector or 2-tuples or ordered pairs and $R^3$ as th or 3 -tuples or ordered triads. Similarly, $R^n$ is the set of all $n$ -vectors or $n$-tuples. Note that following may be taken as a vector: a polynomial, a matrix, a function, a number (real, rational,complex) and a sequence. The following two operations are used in defining the vector space so we firstly study them

1 **Addition of vectors**:

Addition in $R^2$,

If $X=(x_1,x_2)$ and $Y=(y_1,y_2)$ are two vectors in $R^2$, then

$$X+Y=(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2).$$

Addition in $R^3$,

If $X=(x_1,x_2,x_3)$ and $Y=(y_1,y_2,y_3)$ are two vectors in $R^3$, then

$$X+Y=(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3).$$

Addition in $R^n$,

If $X=(x_1,x_2,x_3,\dots,x_n)$ and $Y=(y_1,y_2,y_3,\dots,y_n)$ are two vectors in $R^n$, then

$$X+Y=(x_1,x_2,x_3,\dots,x_n)+(y_1,y_2,y_3,\dots,y_n)=(x_1+y_1,x_2+y_2,x_3+y_3,\dots,x_n+y_n).$$

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2 **Scalar multiplication**

Scalar multiplication in $R^2$ :

Let $c$ be any scalar and $X = (x_1, x_2)$ be any vector in $R^2$ then

$cX = c(x_1, x_2) = (cx_1, cx_2)$

Scalar multiplication in $R^3$ :

If $X = (x_1, x_2, x_3)$ be any vector in $R^3$, then

$cX = c(x_1, x_2, x_3)$

Scalar multiplication in $R^n$ :

If $X = (x_1, x_2, x_3, \dots, x_n)$ be any vector in $R^n$, then

$cX = c(x_1, x_2, x_3, \dots, x_n) = (cx_1, cx_2, cx_3, \dots, cx_n)$

**Definition**:

1 Set $V$ is called a vector space over a field $F$ if the operations "addition of vectors" (denoted by +) and scalar multiplication (denoted by .) are defined such that the following properties hold:

- o is a binary operation, i.e. $X + Y \in V$ for all $X, Y \in V$.
- o is associative, i.e. $(X + Y) + Z = X + (Y + Z)$ for all $X, Y, Z \in V$.
- o is commutative , i.e. $X + Y = Y + X$ for all $X, Y \in V$.

**1 Additive identity**: There exists $O \in V$ such that $O + X = X + O = X$ for all $X \in V$.

**2 Additive inverse:** To each element $X$ of $V$ there is an element $Y$ in $V$ such that $X + Y = O$. Then $Y$ is the additive inverse of $X$.It is denoted by $-X$.

3 Scalar multiplication is well defined i.e. $\forall\ c \in F$ and $X \in V, \Rightarrow cX \in V$

**5 Additive inverse**: To each element $X$ of $V$ there is an element $Y$ in $V$ such that $X + Y = O. Then$ the additive inverse of $X$.It is denoted by $-X$.

6 Scalar multiplication is well defined i.e. $\forall\ c \in F$ and $X \in V, \Rightarrow cX \in V$

7 $c(X + Y) = cX + cY \ \forall\ c \in F$ and $X, Y \in V$

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8 $(c+d)X = cX + dX \; \forall \; c,d \in F$ and $X \in V$

9 $(cd)X = c(dX) \; \forall \; c,d \in F$ and $X \in V$

Scalar multiplication in $R^n$ over the field $R$ :For any scalar $c$ and $X = (x_1, x_2, x_3, \dots, x_n) \in R^n$, we define $cX = (cx_1, cx_2, cx_3, \dots, cx_n)$

Since $cx_1, cx_2, cx_3, \dots, cx_n$ are all real numbers, there $cX \in R^n$ and thus scalar multiplication is well defined. Now, we verify all the axioms of a vector space.

For $X = (x_1, x_2, x_3, \dots, x_n), Y = (y_1, y_2, y_3, \dots, y_n)$ and

$Z = (z_1, z_2, z_3, \dots, z_n)$ in $R^n$ and $c$ in $R$.

1 Addition of vectors is a binary operation: We have

$$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n).$$

Since $x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n$ are all real numbers, therefore

$X + Y \in R^n$.

Thus $R^n$ is closed under the composition addition of vectors.

2. Addition of vector is associative: We have

$$\begin{aligned}
(X+Y)+Z &= [(x_1, x_2, x_3, \dots, x_n) + (y_1, y_2, y_3, \dots, y_n)] + (z_1, z_2, z_3, \dots, z_n) \\
&= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n) + (z_1, z_2, z_3, \dots, z_n) \\
&= ([x_1 + y_1] + z_1, [x_2 + y_2] + z_2, [x_3 + y_3] + z_3, \dots, [x_n + y_n] + z_n) \\
&= (x_1 + [y_1 + z_1], x_2 + [y_2 + z_2], x_3 + [y_3 + z_3], \dots, x_n + [y_n + z_n]) \\
&= (x_1, x_2, x_3, \dots, x_n) + (y_1 + z_1, y_2 + z_2, y_3 + z_3, \dots, y_n + z_n) \\
&= X + (Y + Z)
\end{aligned}$$

Thus the composition "addition of vectors" is associative.

3. Addition of vectors is commutative We write

$$\begin{aligned}
X + Y &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n) \\
&= (y_1 + x_1, y_2 + x_2, y_3 + x_3, \dots, y_n + x_n) \\
&= Y + X
\end{aligned}$$

Thus "addition of vectors" is a commutative composition.

4. Additive identity: We have $O = (0,0,0, \dots ,0) \in R^n$ such that

$$\begin{aligned}
O + X &= (0 + x_1, 0 + x_2, 0 + x_3, \dots ,0 + x_n) \\
&= (x_1, x_2, x_3, \dots, x_n) = X
\end{aligned}$$

for any $X \in R^n$.

Therefore $O = (0,0,0, \dots ,0)$ is the additive identity in $R^n$.

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5 Additive inverse: For $X=(x_1,x_2,x_3,\dots,x_n)\in R^n$, there is an element $-X=(-x_1,$ in $R^n$ such that

$$\begin{aligned} X+(-X) &= (x_1-x_1,x_2-x_2,x_3-x_3,\dots,x_n-x_n) \\ &= (0,0,0,\dots,0) \\ &= O \end{aligned}$$

Therefore $-X$ is the additive inverse of $X$. Thus $R^n$ is an abelian group with respe vectors' composition. Now we try to prove remaining four properties of vector space.

6.

$$\begin{aligned} c(X+Y) &= c(x_1+y_1,x_2+y_2,x_3+y_3,\dots,x_n+y_n) \\ &= (c[x_1+y_1],c[x_2+y_2],c[x_3+y_3],\dots,c[x_n+y_n]) \\ &= (cx_1+cy_1,cx_2+cy_2,cx_3+cy_3,\dots,cx_n+cy_n) \\ &= (cx_1,cx_2,cx_3,\dots,cx_n)+(cy_1,cy_2,cy_3,\dots,cy_n) \\ &= cx+cY \end{aligned}$$

Thus $c(X+Y)=cX+cY$

7.

$$\begin{aligned} (c+d)X &= ([c+d]x_1,[c+d]x_2,[c+d]x_3,\dots,[c+d]x_n) \\ &= (cx_1+dx_1,cx_2+dx_2,cx_3+dx_3,\dots,cx_n+dx_n) \\ &= (cx_1,cx_2,cx_3\dots,cx_n)+(dx_1,dx_2,dx_3,\dots,dx_n) \\ &= cX+dX \end{aligned}$$

Thus $(c+d)X=cX+dX$.

8.

$$\begin{aligned} (cd)X &= ([cd]x_1,[cd]x_2,[cd]x_3,\dots,[cd]x_n) \\ &= (c[dx_1],c[dx_2],c[dx_3],\dots,c[dx_n]) \\ &= c(dx_1,dx_2,dx_3,\dots,dx_n) \\ &= c(dX) \end{aligned}$$

Thus $(cd)X=c(dX)$.

9.

$$\begin{aligned} 1X &= (1x_1,1x_2,1x_3,\dots,1x_n) \\ &= (x_1,x_2,x_3,\dots,x_n) \\ &= X \end{aligned}$$

Thus $1X=X$.

Hence $R^n$ is a vector space over the field $R$.

**Note:**

As of $R^n$, we can show that $R^2,R^3\&R^4$ are vector space over $R$.



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The following are some problems for students to examine whether they are vector spaces or not:

1 The set $V$ of all $m \times n$ matrices with their elements as real numbers with respect to addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication.

2 Let $R$ be the field of real numbers and let $P_n$ be the set of all polynomials of degree at most $n$ over the field $R$.

3 Let $V$ be the set of all pairs $(x, y)$ of real numbers and let $F$ be the field of real numbers with the operations
Addition of vectors $(x_1, y_1) + (x_2, y_2) = (3y_1 + 3y_2, -x_1 - x_2)$
Scalar multiplications $c(x_1, x_2) = (3cx2, -cx_1)$

### 3.1.4 Inner Product Spaces

An inner product space or a Harsdorf pre-Hilbert space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors, often denoted using angle brackets. Inner products allow the rigorous introduction of intuitive geometrical notions, such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension, and are studied in functional analysis. Inner product spaces over the field of complex numbers are sometimes referred to as unitary spaces. The first usage of the concept of a vector space with an inner product is due to Giuseppe Peano, in 1898.

**a) Inner Product Space:**

An inner product on a vector space $V(F)$ is a function that assigns, to every ordered pair of vectors , $u, v \in V$ a scalar in $F$ such that $\forall u, v, w \in V$ and $\forall \alpha, \beta \in F$ the following axioms hold.

$\overline{\langle u, v\rangle} = \langle v, u\rangle$ , where the bar denotes the complex conjugation

$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$

$\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$



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A vector space $V(F)$ with an inner product on it is called an inner product space.

**(a) Norm or length of a vector**

Let be an inner product space. For $v \in V$, we define the norm or length of $v$ by . $\|v\| = \sqrt{\langle v, v\rangle}$

The vector $v$ is called a unit vector if $\|v\| = 1$.

**(b) Orthogonal Vectors**

Let be an inner product space. The vectors $u$ and $v$ in $V$ are orthogonal (or perpendicular) if . $\langle u, v\rangle = 0$

A subset $S$ of $V$ is orthogonal if any two distinct vectors in $S$ are orthogonal.

**Example: 1**

Let $V(F) = R^3(R)$ be a vector space. $\forall u = (a_1, a_2, a_3),\ v = (b_1, b_2, b_3)$ defined by $u = (a_1, a_2, a_3),\ v = (b_1, b_2, b_3)$ and $w = (c_1, c_2, c_3)$, Verify it is an inner product space.

**Solution:**

Let $u = (a_1, a_2, a_3),\ v = (b_1, b_2, b_3)$ and $w = (c_1, c_2, c_3)$

$$(i)\ \overline{\langle u, v\rangle} = \langle u, v\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 [\text{Since its real}]$$
$$= b_1 a_1 + b_2 a_2 + b_3 a_3$$
$$\overline{\langle u, v\rangle} = \langle v, u\rangle$$

$$(ii)\ \langle u, u\rangle = a_1 a_1 + a_2 a_2 + a_3 a_3$$
$$= a_1^2 + a_2^2 + a_3^2$$
$$\langle u, u\rangle > 0\ \forall u \neq 0$$
$$\langle u, u\rangle = 0 \Leftrightarrow a_1^2 + a_2^2 + a_3^2 = 0 \Leftrightarrow a_1 = 0, a_2 = 0, a_3 = 0 \Leftrightarrow u = (0,0,0)$$

$$(iii)\ \alpha u + \beta v = \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3) = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$$
$$\langle \alpha u + \beta v, w\rangle = (\alpha a_1 + \beta b_1)c_1 + (\alpha a_2 + \beta b_2)c_2 + (\alpha a_3 + \beta b_3)c_3$$
$$= \alpha(a_1 c_1 + a_2 c_2 + a_3 c_3) + \beta(b_1 c_1 + b_2 c_2 + b_3 c_3)$$
$$\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$$

$\therefore V = R^3(R)$ is an inner product space.

**Example:2**

Let $u = (a_1, a_2, a_3, \ldots, a_n),\ v = (b_1, b_2, b_3, \ldots, b_n) \in F^n(C)$. Define .

$\langle u, v\rangle = a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3} + \ldots + a_n\overline{b_n}$ Verify that it is an inner product of $F^n(C)$.



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**Solution:**

$$(i)\overline{\langle u,v\rangle}=\overline{a_1\overline{b_1}+a_2\overline{b_2}+a_3\overline{b_3}+\ldots+a_n\overline{b_n}}$$
$$=\overline{a_1\overline{b_1}}+\overline{a_2\overline{b_2}}+\overline{a_3\overline{b_3}}+\ldots+\overline{a_n\overline{b_n}}$$
$$=\overline{a_1}b_1+\overline{a_2}b_2+\overline{a_3}b_3+\ldots+\overline{a_n}b_n$$
$$=b_1\overline{a_1}+b_2\overline{a_2}+b_3\overline{a_3}+\ldots+b_n\overline{a_n}$$
$$=\langle v,u\rangle$$

$$(ii)\langle u,u\rangle=a_1\overline{a_1}+a_2\overline{a_2}+a_3\overline{a_3}+\ldots+a_n\overline{a_n}$$
$$\langle u,u\rangle=|a_1|^2+|a_2|^2+|a_3|^2+\ldots+|a_n|^2\geq 0 \text{ if } u\neq 0$$
$$\langle u,u\rangle=0\Leftrightarrow u=0$$

$(iii)$ Let $w=(c_1,c_2,c_3,\ldots,c_n)$

$$\alpha u+\beta v=\alpha(a_1,a_2,a_3,\ldots,a_n)+\beta(b_1,b_2,b_3,\ldots,b_n)=(\alpha a_1+\beta b_1,\alpha a_2+\beta b_2,\alpha a_3+\beta b_3,\ldots,\alpha a_n+\beta b_n)$$
$$\langle\alpha u+\beta v,w\rangle=(\alpha a_1+\beta b_1)\overline{c_1}+(\alpha a_2+\beta b_2)\overline{c_2}+(\alpha a_3+\beta b_3)\overline{c_3}+\ldots.+(\alpha a_n+\beta b_n)\overline{c_n}$$
$$=\alpha(a_1\overline{c_1}+a_2\overline{c_2}+a_3\overline{c_3}+\ldots+a_n\overline{c_n})+\beta(b_1\overline{c_1}+b_2\overline{c_2}+b_3\overline{c_3}+\ldots+b_n\overline{c_n})$$
$$\langle\alpha u+\beta v,w\rangle=\alpha\langle u,w\rangle+\beta\langle v,w\rangle$$

Hence $\langle u,v\rangle$ is an inner product space on $F^n(C)$

**Example:3**

Let V be the set of all continuous real functions defined on the closed interval [0,1]. The inner product on V defined by $\langle f(x),g(x)\rangle=\int_0^1 f(t)\,g(t)\,dt$. Prove that V(R) is an inner product space.

**Solution:**

$$(i)\overline{\langle f,g\rangle}=\int_0^1\overline{f(t)\,g(t)}\,dt=\int_0^1\overline{g(t)}\,\overline{f(t)}\,dt=\int_0^1 g(t)f(t)\,dt=\langle g,f\rangle$$

$$(ii)\ \langle f,f\rangle=\int_0^1 f(t)\,f(t)\,dt=\int_0^1\left(f(t)\right)^2dt\geq 0,\ f(t)\neq 0$$

and $\langle f,f\rangle=0$ iff $f(t)=0,\forall t\in[0,1]$

$$(iii)\langle\alpha f+\beta g,h\rangle=\int_0^1\left(\alpha f(t)+\beta\,g(t)\right)h(t)\,dt$$
$$=\alpha\int_0^1 f(t)h(t)\,dt+\beta\int_0^1 g(t)h(t)\,dt$$
$$\langle\alpha f+\beta g,h\rangle=\alpha\langle f,h\rangle+\beta\langle g,h\rangle$$

$\therefore\langle f,g\rangle$ is an inner product space over $R$

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**Example: 4**

Prove that $R^2(R)$ is an inner product space with the inner product defined by $\langle u,v\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2, \quad \forall u=(a_1,a_2), v=(b_1,b_2)$.

**Solution:**

$$
\begin{aligned}
(i)\,\overline{\langle u,v\rangle} &= \overline{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}\\
&= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2\\
&= b_1a_1 - b_1a_2 - b_2a_1 + 2b_2a_2\\
&= \langle v,u\rangle
\end{aligned}
$$

$$
\begin{aligned}
(ii)\langle u,u\rangle &= a_1a_1 - a_2a_1 - a_1a_2 + 2a_2a_2\\
&= a_1^2 - 2a_1a_2 + a_2^2 + a_2^2
\end{aligned}
$$

$$\langle u,u\rangle = (a_1-a_2)^2 + a_2^2 \geq 0 \ \text{ if } u \neq 0$$

$$\langle u,u\rangle = 0 \,\text{iff}\, (a_1-a_2)^2 + a_2^2 = 0 \Leftrightarrow a_1 - a_2 = 0, a_2 = 0 \Leftrightarrow a_1 = a_2 = 0 \Leftrightarrow u = 0$$

$(iii)$ Let $w=(c_1,c_2),\ \alpha,\beta \in R$

$$\alpha u + \beta v = \alpha(a_1,a_2) + \beta(b_1,b_2) = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2)$$

$$
\begin{aligned}
\langle \alpha u + \beta v, w\rangle &= \langle(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2),(c_1,c_2)\rangle\\
&= (\alpha a_1 + \beta b_1)c_1 - (\alpha a_2 + \beta b_2)c_1 - (\alpha a_1 + \beta b_1)c_2 + 2(\alpha a_2 + \beta b_2)c_2\\
&= \alpha(a_1c_1 - a_2c_1 - a_1c_2 + 2a_2c_2) + \beta(b_1c_1 - b_2c_1 - b_1c_2 + 2b_2c_2)
\end{aligned}
$$

$$\langle \alpha u + \beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$$

Hence $R^2(R)$ is an inner product space

### 3.1.5 Theorems

**Theorem: 1**

Let V be an inner product space. Then for $\forall u,v,w \in V$ and $\alpha \in F$, the following statements are true.

$(i)$ $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

$(ii)$ $\langle u, \alpha v\rangle = \overline{\alpha}\langle u,v\rangle$

$(iii)$ $\langle u,0\rangle = \langle 0,u\rangle = 0$

$(iv)$ If $\langle u,v\rangle = \langle u,w\rangle, \forall u \in V$ then $v = w$

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**Proof:**

$$(i)\ \langle u, v+w\rangle = \overline{\langle v+w, u\rangle}$$
$$= \overline{\langle v,u\rangle + \langle w,u\rangle}$$
$$= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle}$$
$$\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$$

$$(ii)\ \langle u, \alpha v\rangle = \overline{\langle \alpha v, u\rangle}$$
$$= \overline{\alpha}\,\overline{\langle v,u\rangle}$$
$$\langle u, \alpha v\rangle = \overline{\alpha}\langle u,v\rangle$$

$$(iii)\langle u,0\rangle = \langle u,0v\rangle = \overline{0}\langle u,v\rangle = 0$$
$$\langle 0,u\rangle = \langle 0v,u\rangle = 0\langle v,u\rangle = 0$$
$$\therefore \langle u,0\rangle = \langle 0,u\rangle = 0$$

$$(iv)\ \text{Consider, } \langle u, v-w\rangle = \langle u,v\rangle - \langle u,w\rangle \text{ using(i)}$$
$$=0 \text{ (given)}$$
$$\Rightarrow v-w=0 \text{ using(iii)}$$
$$\Rightarrow v=w$$

**Theorem: 2**

Let V be an inner product space over F, then $\forall u,v\in V$ and $\alpha,\beta\in F$ we have

$(i)\ \|\alpha u\| = |\alpha|\|u\|$

$(ii)|\langle u,v\rangle| \le \|u\|\|v\|$ (*Cauchy Schwartz inequality*)

$(iii)\|u+v\| \le \|u\|+\|v\|$ (*Triangular inequality*)

**Proof:**

$$(i)\ \|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle$$
$$= \alpha\langle u, \alpha u\rangle$$
$$= \alpha\overline{\langle \alpha u, u\rangle}$$
$$= \alpha\overline{\alpha}\overline{\langle u,u\rangle}$$
$$= \alpha\overline{\alpha}\langle u,u\rangle$$
$$\|\alpha u\|^2 = |\alpha|^2\|u\|^2$$
$$\Rightarrow \|\alpha u\| = |\alpha|\|u\|$$



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$(ii)$ Cauchy Schwartz Inequality

Case (i)

If $u=0$ (or) $v=0$ then $\langle u,v\rangle=0$

Also $\|u\|=0, \|v\|=0$

Hence $\langle u,v\rangle=\|u\|\|v\|$

Case(ii)

Let $u\neq 0$ and $v\neq 0$.

Let $w=v-\frac{\langle v,u\rangle}{\|u\|^2}u$

since $\frac{\langle v,u\rangle}{\|u\|^2}\in F$, let $\frac{\langle v,u\rangle}{\|u\|^2}=k$

$\therefore\ w=v-ku$

$$\begin{aligned}
\text{consider } \langle w,w\rangle &= \langle v-ku, v-ku\rangle \\
&= \langle v,v\rangle-\langle v,ku\rangle-\langle ku,v\rangle+\langle ku,ku\rangle \\
&= \|v\|^2-\bar{k}\langle v,u\rangle-k\langle u,v\rangle+k\bar{k}\|u\|^2 \\
&= \|v\|^2-\frac{\overline{\langle v,u\rangle}}{\|u\|^2}\langle v,u\rangle-\frac{\langle v,u\rangle}{\|u\|^2}\langle u,v\rangle+|k|^2\|u\|^2 \\
&= \|v\|^2-\frac{\langle u,v\rangle\overline{\langle u,v\rangle}}{\|u\|^2}-\frac{\overline{\langle u,v\rangle}\langle u,v\rangle}{\|u\|^2}+\frac{|\langle v,u\rangle|^2}{\|u\|^4} \\
&= \|v\|^2-\frac{|\langle u,v\rangle|^2}{\|u\|^2}-\frac{|\langle u,v\rangle|^2}{\|u\|^2}+\frac{|\langle u,v\rangle|^2}{\|u\|^2}
\end{aligned}$$

$$\langle w,w\rangle=\|v\|^2-\frac{|\langle u,v\rangle|^2}{\|u\|^2}$$

We know that, $\langle w,w\rangle\geq 0$

$$\Rightarrow \|v\|^2-\frac{|\langle u,v\rangle|^2}{\|u\|^2}\geq 0$$

$$\|v\|^2\geq\frac{|\langle u,v\rangle|^2}{\|u\|^2}$$

$$\|u\|^2\|v\|^2\geq|\langle u,v\rangle|^2$$

$$|\langle u,v\rangle|\leq\|u\|\|v\|$$

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$(iii)$ Triangular Inequality

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 2\,\mathrm{Re}\langle u,v\rangle + \|v\|^2 \\
&= \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \quad [\text{since}, 2\,\mathrm{Re}(z) = 2|z|] \\
&\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad [\text{since}, |\langle u,v\rangle| \le \|u\|\|v\|] \\
\|u+v\|^2 &\le (\|u\| + \|v\|)^2 \\
\Rightarrow \|u+v\| &\le \|u\| + \|v\|
\end{aligned}$$

**Theorem: 3**

In an inner product space V(R), $\forall x, y \in V$,

$(i)$ $\big|\,\|x\| - \|y\|\,\big| \le \|x-y\|$

$(ii)$ $\|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle$

**Proof:**

$(i)$
$$\begin{aligned}
\|x\| &= \|(x-y)+y\| \\
\|x\| &\le \|x-y\| + \|y\| \\
\|x\| - \|y\| &\le \|x-y\| \quad \text{------}(1) \\
\|y\| &= \|(y-x)+x\| \\
\|y\| &\le \|y-x\| + \|x\| \\
\|y\| - \|x\| &\le \|y-x\| \\
-(\|x\| - \|y\|) &\le \|x-y\| \quad \text{------}(2)
\end{aligned}$$

from (1) & (2),

$\big|\,\|x\| - \|y\|\,\big| \le \|x-y\|$

$(ii)$
$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\
&= \langle x,y\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,x\rangle \\
\|x+y\|^2 - \|x-y\|^2 &= 4\langle x,y\rangle \quad [\text{since V is real}, \langle x,y\rangle = \langle y,x\rangle\ ]
\end{aligned}$$

**Theorem: 4**

In an inner product space V, any subset of non-zero orthogonal vectors are linearly independent.

**Proof:**



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Let $S = \{v_1, v_2, v_3, ..., v_n\}$ be a set of orthogonal vectors, which are non-zero.

$\therefore \forall v_i \neq 0,\ \langle v_i, v_j \rangle = 0 \ if\ \ i \neq j$

Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ... + \alpha_n v_n = 0.$

Consider $\langle \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ... + \alpha_n v_n, v_i \rangle = 0 \ [\because \langle 0, v_i \rangle = 0$ by theorem1$]$

$\Rightarrow \alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \alpha_3 \langle v_3, v_i \rangle + ... + \alpha_n \langle v_n, v_i \rangle = 0$

$\Rightarrow \alpha_1.0 + \alpha_2.0 + \alpha_3.0 + ... + \alpha_n.0 = 0$

$\Rightarrow \alpha_i \langle v_i, v_i \rangle = 0$

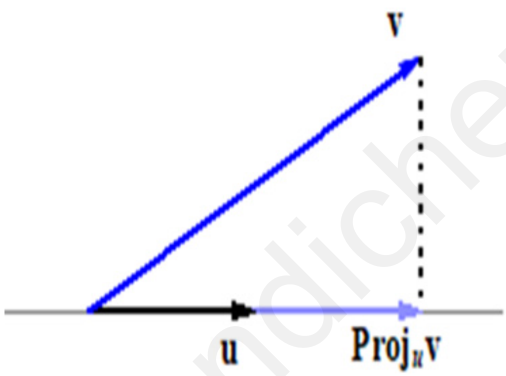
$\Rightarrow \alpha_i \|v_i\|^2 = 0$

$\Rightarrow \alpha_i = 0$ Hence S is linearly independent.

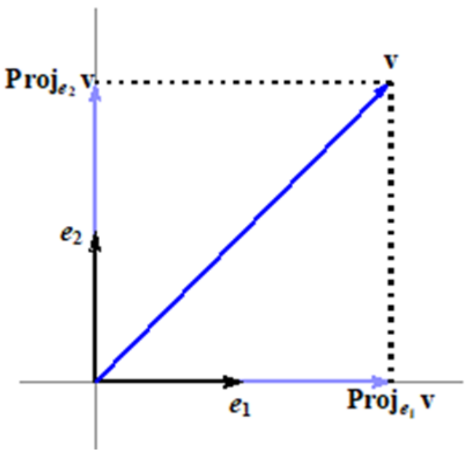
### 3.1.6 Orthogonality

**Orthogonal Projection**

The idea of orthogonal projection is best depicted in the following figure.




The orthogonal projection of v onto u gives the component vector Proj$_u$v of v in the direction of u. This fact is best demonstrated in the case that u is one of the standard basis vectors.



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As shown in the figure above, the lengths of the orthogonal projections in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions, respectively, give the coordinates of the vector **v** in the standard basis. On the other hand, each coordinate can be obtained by computing the dot product of **v** and the corresponding standard basis vector, *i.e.*,

$$\text{IProj}_{e1}\text{vI} = \text{v} \cdot \text{e}_1, \text{ and } \text{IProj}_{e2}\text{vI} = \text{v} \cdot \text{e}_2.$$

However, the orthogonal projection of v in the $e_1$ direction should not depend on the length of the vector we use to specify the direction. Hence, the validity of the observation above is based on the fact that $e_1$ and $e_2$ are "special" in some sense. The observation holds true precisely because the vectors $e_1$ and $e_2$ are unit vectors.

To obtain a similar conclusion in the general setting, consider vectors u and v in the first figure. We first normalize u to get

$$\hat{u} = \frac{u}{\sqrt{u.u}}$$

Now, this unit vector $\hat{u}$ satisfies that $\left\| \mathrm{Proj}_{\hat{u}} v \right\| = v.\hat{u}$

Because u and $\mathrm{Proj}_{\hat{u}} v$ are in the same direction, we have $\mathrm{Proj}_{\hat{u}} v = \mathrm{Proj}_u v.$. Thus

$$\mathrm{Proj}_u v = \left\| \mathrm{Proj}_{\hat{u}} v \right\| \hat{u} = (v.\hat{u})\hat{u} = \left( \frac{v.u}{\sqrt{u.u}} \right) \left( \frac{u}{\sqrt{u.u}} \right) = \frac{v.u}{u.u} u$$

**Definition:**

Given vectors u, v ∈ Rm$^n$, where $u \neq 0$, then the **orthogonal projection** of **v** onto **u** is defined to be

$$\mathrm{P} = \mathrm{Proj}_u v = \frac{v.u}{u.u} u$$

Let V be an inner product space. Let $x, v \in V$, $v \neq 0$. Then

$$\mathrm{P} = \frac{\langle x, v \rangle}{\langle v, v \rangle} v$$

is the **orthogonal projection** of the vector x onto the vector v.

If $v_1, v_2, \ldots, v_n$ is an orthogonal set of vectors, then

$$\mathrm{P} = \mathrm{Proj}_u v = \frac{\langle x.v_1 \rangle}{\langle v_1.v_1 \rangle} v_1 + \frac{\langle x.v_2 \rangle}{\langle v_2.v_2 \rangle} v_2 + \ldots + \frac{\langle x.v_n \rangle}{\langle v_n.v_n \rangle} v_n$$

is the **orthogonal projection** of the vector x onto the subspace spanned by

$v_1, v_2, \ldots, v_n$

### 3.1.7 Gram Schmidt Orthogonalization Process

**Orthonormal basis:**

Let $V$ be an inner product space. A subset $S$ of $V$ is orthonormal basis if it is an ordered basis that is orthonormal.

**Example:** The set $\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$ is an orthonormal basis for $R^2$.

**Theorem:1 (Gram Schmidt Orthogonalization Process)**

Every finite dimensional inner product space has an orthonormal set as a basis.

**Proof:**

Let V(F) be a finite dimensional inner product space and $\dim(V) = n$.

Let $B = \{v_1, v_2, v_3, \ldots, v_n\}$ be a basis for V(F).

**Claim:** we have to construct an orthonormal basis $\{w_1, w_2, w_3, \ldots, w_n\}$ from $B$

First we shall construct an orthogonal basis $\{u_1, u_2, u_3, \ldots, u\}$ from $B$.

We prove by induction on $n$.

Take $u_1 = v_1 \neq 0$

Let $u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1$, We have to prove $u_2 \neq 0$

For, if $v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = 0$ then $v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = 0$

$$\Rightarrow v_2 = \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1$$

$\Rightarrow v_2, u_1$ are dependent, which is a contradiction

$\therefore \mathrm{u}_2 \neq 0$

**Claim:** $u_2$ is orthogonal to $u_1$

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$$\langle u_2, u_1 \rangle = \langle v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1, u_1 \rangle$$

$$= \langle v_2, u_1 \rangle - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \langle u_1, u_1 \rangle$$

$$= \langle v_2, u_1 \rangle - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \|u_1\|^2$$

$$= \langle v_2, u_1 \rangle - \langle v_2, u_1 \rangle$$

$$\langle u_2, u_1 \rangle = 0$$

$\Rightarrow u_2$ is orthogonal to $u_1$

$\Rightarrow \{u_1, u_2\}$ is an orthogonal set

Hence the theorem is true for $n = 2$.

Now, Assume the theorem is true for all integers up to $\{u_1, u_2, u_3, ...$ .

i.e. $\{u_1, u_2, u_3, ..., u_k\}$ is an orthogonal set.

Now, we prove the theorem for $n = k+1$

Let $u_{k+1} = v_{k+1} - \frac{\langle v_{k+1}, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_{k+1}, u_2 \rangle}{\|u_2\|^2} u_2 - ... - \frac{\langle v_{k+1}, u_k \rangle}{\|u_k\|^2} u_k$ then $u_{k+1} \neq 0$

If $u_{k+1} = 0$, then $v_{k+1} = \frac{\langle v_{k+1}, u_1 \rangle}{\|u_1\|^2} u_1 + \frac{\langle v_{k+1}, u_2 \rangle}{\|u_2\|^2} u_2 + ... + \frac{\langle v_{k+1}, u_k \rangle}{\|u_k\|^2} u_k$

is a linear combination of $\{u_1, u_2, u_3, ..., u_k\}$ and hence linear combination of $\{v_1, v_2, v_3, ..., v_k\}$, which is a contradiction? Hence $u_{k+1} \neq 0$.

**To prove:** $u_1, u_2, u_3, ..., u_k$ is orthogonal to $u_1, u_2, u_3, ..., u_k$. i.e. $\langle u_{k+1}, u_i \rangle = 0$

$$\langle u_{k+1}, u_i \rangle = \langle v_{k+1} - \frac{\langle v_{k+1}, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_{k+1}, u_2 \rangle}{\|u_2\|^2} u_2 - ... - \frac{\langle v_{k+1}, u_k \rangle}{\|u_k\|^2} u_k, u_i \rangle$$

$$= \langle v_{k+1}, u_i \rangle - \frac{\langle v_{k+1}, u_1 \rangle}{\|u_1\|^2} \langle u_1, u_i \rangle - .... - \frac{\langle v_{k+1}, u_k \rangle}{\|u_k\|^2} \langle u_k, u_i \rangle$$

$$\langle u_{k+1}, u_i \rangle = 0$$

Hence $\{u_1, u_2, u_3, ..., u_{k+1}\}$ is an orthogonal set.

Hence the theorem is true for n.

Hence, every finite dimensional inner product space has an orthonormal set as a basis.

**Theorem: 2**

Let V be an inner product space and $S = \{v_1, v_2, v_3, ..., v_n\}$ be an orthogonal subset of V consisting of

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non-zero vectors. If $v \in L(S)$, then $S = \{v_1, v_2, v_3, \ldots, v_n\}$.

**Proof:**

Given: V is an inner product space over F.

$S = \{v_1, v_2, v_3, \ldots, v_n\}$ is a subset of V.

Let $v \in L(S)$, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n$

$$\begin{aligned}
\langle v, v_i \rangle &= \langle \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n, v_i \rangle \\
&= \alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \alpha_3 \langle v_3, v_i \rangle + \ldots + \alpha_i \langle v_i, v_i \rangle + \ldots + \alpha_n \langle v_n, v_i \rangle \\
&= \alpha_i \langle v_i, v_i \rangle \quad [\text{since } \langle v_i, v_j \rangle = 0 \ if \ i \neq j]
\end{aligned}$$

$$\langle v, v_i \rangle = \alpha_i \|v_i\|^2$$

Where, $\alpha_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$, $i = 1, 2, 3, \ldots$

$$V = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

**Example 1:**

Let $u = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Describe the set of all vectors in $\mathbb{R}^3$ that are orthogonal to both u and v

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a be a vector orthogonal to both u and v. Then $x.u = 0$ and $x.v = 0$.

Thus,

$$\begin{aligned}
x_1 + 5x_1 &= 0 \\
-x_1 + x_3 &= 0
\end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

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$$\Rightarrow x_1 = x_3$$
$$x_2 = -\frac{1}{5}x_3$$
$$x_3 = x_3$$

Thus, $x = x_3 = \begin{bmatrix} 1 \\ -\frac{1}{5} \\ 1 \end{bmatrix}$, where $x_3$ is a real number.

The vectors in orthogonal to both u and v are the scalar multiples of $\begin{bmatrix} 1 \\ -\frac{1}{5} \\ 1 \end{bmatrix}$

**Example 2:**

Let y=$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Let $L = Span\{u\}$

(a) Find the orthogonal projection of y onto L.

(b) Write y as a sum of a vector in L and a vector orthogonal to L

**Solution:**

$$(a)\, \vec{y} = proj_L y = \left(\frac{y.u}{u.u}\right)u = -\frac{13}{65}\begin{bmatrix} 4 \\ -7 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$(b)\;\; y = \vec{y} + z$$

$$\vec{y} = \begin{bmatrix} -\frac{4}{5} \\ \frac{7}{5} \end{bmatrix}$$

$$z = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{4}{5} \\ \frac{7}{5} \end{bmatrix} \qquad = \begin{bmatrix} \frac{14}{5} \\ \frac{8}{5} \end{bmatrix}$$

**Example 3:**

Let $x = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ and $W = span\{u, v\}$. Note that $u \cdot v = 0$ .Find a vector $a$ in $W$ ,such that $x = a + b$ .

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**Solution**:

$$a = proj_W x = \left(\frac{x\cdot u}{u\cdot u}\right)u + \left(\frac{x\cdot v}{v\cdot v}\right)v,\ b = x - a$$

$$a = \frac{-2}{6}\begin{bmatrix}2\\-1\\1\end{bmatrix} + \frac{28}{21}\begin{bmatrix}-1\\2\\4\end{bmatrix} = \frac{-1}{3}\begin{bmatrix}2\\-1\\1\end{bmatrix} + \frac{4}{3}\begin{bmatrix}-1\\2\\4\end{bmatrix} = \begin{bmatrix}-2\\3\\5\end{bmatrix}, b = x - a = \begin{bmatrix}2\\3\\-1\end{bmatrix}$$

**Example 4**:

Given a vector $x = \begin{bmatrix}2\\4\end{bmatrix}$ and a line $L = \left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} : 4x_1 + 3x_2 = 0\right\}$ in $R^2$.

(a) Find the vector in L that is closest to x that is, find the orthogonal projection of x onto the line L.

Solution:

$$L = span(u) \text{ where u=} \begin{bmatrix}-3\\4\end{bmatrix}$$

we have that

$$(a) x = proj_L x = \left(\frac{x\cdot u}{u\cdot u}\right)u = \frac{10}{25}\begin{bmatrix}-3\\4\end{bmatrix} = \frac{2}{5}\begin{bmatrix}-3\\4\end{bmatrix} = \begin{bmatrix}-\frac{6}{5}\\\frac{8}{6}\end{bmatrix}$$

**Example 5:**

In ,letthesubspace$S$bethespanofthevectors $\vec{v_1} = (1,1,-1,-1)$ and $\vec{v_2} = (1,1,1,1)$.

Find the orthogonal projection of $\vec{x} = (1,2,3,4)$ into $S$.

**Solution:**

Note that the vectors $\vec{v_1}$ and $\vec{v_2}$ are an orthogonal basis for S. We want to write $\vec{x} = a_1\vec{v_1} + a_2\vec{v_2} + \vec{w}$ where $\vec{w} \perp S$. (1)

Then the orthogonal projection of $\vec{x}$ onto $S$ will be

$$P_S\vec{x} = a_1\vec{v_1} + a_2\vec{v_2}$$

By the general strategy use above, to find $a_1$ take the inner product of both sides of the equation (1) with $\vec{v_2}$ .Because $\vec{v_2}$ is orthogonal to both $\vec{v_2}$ and $w$ , we obtain

$$\langle \vec{x_1}, \vec{v_1}\rangle = a_1\langle \vec{v_1}, \vec{v_1}\rangle \text{ so } a_1 = \frac{\langle \vec{x_1}, \vec{v_1}\rangle}{\left\|\vec{v_1}^2\right\|} = \frac{-4}{4} = -1.$$

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Similarly, $a_2 = \frac{\langle \vec{x}_1, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} = \frac{10}{4} = \frac{5}{2}$

Using these values in equation number (1) we find the projection of $\vec{x}$ into S is

$$P_S\vec{x} = -1\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}_1 \vec{v}_1 + \frac{5}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \vec{v}_2 = \frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 7 \\ 7 \end{bmatrix}$$

The projection of $\vec{x}$ into S is

$$\vec{w} = P_s \perp \vec{x} = \vec{x} - P_s\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3 \\ 3 \\ 7 \\ 7 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

**3.1.8 Linear Combination**-Linear and convex combinations of vectors

**Definition:**

Let $V$ be a vector space and $S$ be a non empty subset of $V$. A vector $v \in V$ is called a linear combination of vectors of $S$, if there exists $u_1, u_2, u_3, \ldots u_n \in S$; $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in F$ such that $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots \alpha_n u_n$.

**Note:** In any vector space $V$, $0v = 0$ for each $v \in V$. Thus zero vector is a linear combination of any non-empty subset of $V$.

**Linear span:**

Let $S$ be a non empty subset of the vector space $V(F)$. The set of all linear combinations of finite sets of elements of $S$ is called the linear span of $S$ and it is denoted as $L(S)$ or $span(S)$.

A subset $S$ of a vector space $V$ generates (or spans) $V$ if $span(S) = V$.

**Example:1**

In $R^3$ over $R$, test whether $(2, -5, 4)$ is a linear combination of the vectors $(1. -3. 2)$ and $(2, -1, 1)$.

**Solution:**

Let $\alpha, \beta, \in R$

$$(2, -5, 4) = \alpha(1, -3, 2) + \beta(2, -1, 1)$$
$$= (\alpha, -3\alpha, 2\alpha) + (2\beta, -\beta, \beta)$$
$$\therefore (2, -5, 4) = (\alpha + 2\beta, -3\alpha - \beta, 2\alpha + \beta)$$

Therefore, the equations are



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$$\alpha + 2\beta = 2 \qquad \text{.....(1)}$$
$$-3\alpha - \beta = -5 \qquad \text{.....(2)}$$
$$2\alpha + \beta = 4 \qquad \text{.....(3)}$$

We write the above system of equations in the matrix form,

$$\begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$$

Changing into the row echelon form,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -3 & -1 & -5 \\ 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -5 & 1 \\ 0 & -3 & 0 \end{bmatrix} \quad \text{since}$$

$R_2 \to R_2 + 3R_1$ *and* $R_3 \to R_3 - 2R_1$

We get $\alpha + 2\beta = 2, -5\beta = 1$ and $-3\beta = 0$

Here the equations are inconsistent.

$\therefore$ $(2, -5, 4)$ cannot be written as a linear combination of the vectors $(1. -3.2)$ and $(2. -1.1)$.

**Example:2**

In $R^3$ over $R$, test whether $(1, -2, 5)$ is a linear combination of the vectors $(1,1,1), (1,2,3), (2,-1,1)$.

**Solution:**

Let $\alpha, \beta, \gamma \in R$

$$(1, -2, 5) = \alpha(1,1,1) + \beta(1,2,3) + \gamma(2,-1,1)$$
$$= (\alpha, \alpha, \alpha) + (\beta, 2\beta, 3\beta) + (2\gamma, -\gamma, \gamma)$$
$$(1, -2, 5) = (\alpha + \beta + 2\gamma), (\alpha + 2\beta - \gamma), (\alpha + 3\beta + \gamma)$$

Therefore, the equations are

$$\alpha + \beta + 2\gamma = 1 \qquad \text{.....(1)}$$
$$\alpha + 2\beta - \gamma = -2 \qquad \text{.....(2)}$$
$$\alpha + 3\beta + \gamma = 5 \qquad \text{.....(3)}$$

We write the above system of equations in the matrix form,

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Changing into the row echelon form,

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix}$$



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$$□\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \quad \text{since } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$□\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix} \quad \text{since } R_3 \to R_3 - 2R_1$$

Now the equations are $\alpha + \beta + 2\gamma = 1$ .....(4)

$\beta - 3\gamma = -3$ ..................(5)

and $5\gamma = 10$ .................(6)

From the equations (4), (5) and (6), we get

$\gamma = 2$, $\beta = 3$ and $\alpha = -6$

Substituting $\alpha, \beta, \gamma$ satisfies the equations (1), (2) and (3).

Hence the equations are consistent.

$\therefore$ $(1, -2{,}5)$ is a linear combination of the vectors $(1{,}1{,}1), (1{,}2{,}3), (2, -1{,}1)$

. That is $(1, -2{,}5) = -6(1{,}1{,}1) + 3(1{,}2{,}3) + 2(2, -1{,}1)$.

**Example:3**

In $P_3(R)$ is vector space of polynomials of degree $\leq 3$ over R. Test whether $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$.

**Solution:**

Let $\alpha, \beta \in R$

$\therefore 2x^3 - 2x^2 + 12x - 6 = \alpha(x^3 - 2x^2 - 5x - 3) + \beta(3x^3 - 5x^2 - 4x - 9)$

$\Rightarrow 2x^3 - 2x^2 + 12x - 6 = (\alpha + 3\beta)x^3 + (-2\alpha - 5\beta)x^2 + (-5\alpha - 4\beta)x + (-3\alpha - 9\beta)$

Now comparing $x^3, x^2, x$ and constant terms on both sides, we get

The equations are $\alpha + 3\beta = 2$ .....(1)

$-2\alpha - 5\beta = -2$ .....(2)

$-5\alpha - 4\beta = 12$ .....(3)

and $-3\alpha - 9\beta = -6$ .....(4)

We write the above system of equations in the matrix form,

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$$\begin{bmatrix} 1 & 3 \\ -2 & -5 \\ -5 & -4 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix}$$

Changing into the row echelon form,

$$\square \begin{bmatrix} 0 & 1 & 2 \\ 0 & 11 & 22 \\ 0 & 0 & 0 \end{bmatrix} \square \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 11 & 22 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\square \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 11 & 22 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + 5R_1,\ R_4 \to R_4 + 3R_1$$

$$\square \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - 11R_1$$

Now the equations are $\alpha + 3\beta = 2$ .....(4)

$\beta = 2$ .....(5)

From the equations (4), (5) we get $\beta = 2$ *and* $\alpha = -4$.

Hence the equations are consistent.

$2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$.

$\therefore 2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9)$

.

**Example:4**

Test whether the indicated vector $2, -1,1$ is in the linear span of $S = \{(1,0,2), (-1,1,1)\}$ in $R^3(R)$.

**Solution:**

Given $S = \{(1,0,2), (-1,1,1)\}$

$\therefore L(S) = \{\alpha(1,0,2) + \beta(-1,1,1);\ \alpha, \beta \in R\}$

Let, $(2, -1,1) = \alpha(1,0,2) + \beta(-1,1,1)$

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$$(2,-1,1) = (\alpha - \beta, \beta, 2\alpha + \beta)$$

$\therefore \alpha - \beta = 2,\ \beta = -1\ \ and\ 2\alpha + \beta = 1$

Solving the above equations, we get $\alpha = 2,\ \beta = -1$ which satisfies all the above

equations. Hence the equations are consistent.

$\therefore (2,-1,1) = -1(1,0,2) + 2(-1,1,1)$

$\therefore (2,-1,1) \in L(S)$.

**Example:5**

Test whether the indicated matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in the linear span of

$S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ in $M_2(R)$.

**Solution:**

Given $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

$$L(S) = \left\{ \alpha \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \alpha, \beta, \gamma \in R \right\}$$

If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in L(S)$, then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha + \gamma & \beta + \gamma \\ -\alpha & \beta \end{bmatrix}$$

$\Rightarrow \alpha + \gamma = 1,\ \beta + \gamma = 0,\ -\alpha = 0\ and\ \ \beta = 1$

$\Rightarrow \alpha = 0,\ \beta = 1\ and\ \gamma = -1$

These values are not satisfied the above equations. Hence the equations are inconsistent.

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin L(S)$.

**Example:6**

Show that the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ generates $(M_{2\times2}R)$.

**Solution:**

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (M_{2\times2}R)$

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$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_4 \\ \alpha_1 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = a \qquad \ldots(1)$$
$$\alpha_1 + \alpha_2 + \alpha_4 = b \qquad \ldots(2)$$
$$\alpha_1 + \alpha_3 + \alpha_4 = c \qquad \ldots(3)$$
$$\alpha_2 + \alpha_3 + \alpha_4 = d \qquad \ldots(4)$$

We write the above system of equations in the matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Changing into the row echelon form,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 1 & 1 & 0 & 1 & b \\ 1 & 0 & 1 & 1 & c \\ 0 & 1 & 1 & 1 & d \end{bmatrix}$$

$$□ \begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 0 & 0 & -1 & 1 & b-a \\ 0 & -1 & 0 & 1 & c-a \\ 0 & 1 & 1 & 1 & d \end{bmatrix} \quad \text{since } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

Interchanging the rows $R_3 \to R_2$

$$□ \begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 0 & -1 & 0 & 1 & c-a \\ 0 & 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & 1 & d \end{bmatrix}$$

$$□ \begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 0 & -1 & a-c \\ 0 & 0 & 1 & -1 & a-b \\ 0 & 1 & 1 & 1 & d \end{bmatrix} \quad \text{since } R_2 \to (-1)\times R_2, R_3 \to (-1)\times R_3$$

$$□ \begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 0 & -1 & a-c \\ 0 & 0 & 1 & -1 & a-b \\ 0 & 0 & 1 & 2 & -a+c+d \end{bmatrix} \quad \text{Since } R_4 \to R_4 - R_2$$

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$$\square\begin{bmatrix} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 0 & -1 & a-c \\ 0 & 0 & 1 & -1 & a-b \\ 0 & 0 & 0 & 3 & -2a+b+c+d \end{bmatrix} \quad \text{Since } R_4 \to R_4 - R_3$$

Now the corresponding equations are

$$\alpha_1 + \alpha_2 + \alpha_3 = a \qquad \ldots(4)$$
$$\alpha_2 - \alpha_4 = a - c \qquad \ldots(5)$$
$$\alpha_3 - \alpha_4 = a - b \qquad \ldots(6)$$
$$3\alpha_4 = -2a + b + c + d \qquad \ldots(7)$$
$$\Rightarrow \alpha_4 = \frac{1}{3}(-2a + b + c + d)$$

Substituting $\alpha_4$ in equations (4), (5), (6)

$$\alpha_3 = \frac{1}{3}(a - 2b + c + d), \alpha_2 = \frac{1}{3}(a + b - 2c + d) \text{ and}$$
$$\alpha_1 = \frac{1}{3}(a + b + c - 2d)$$

Which satisfies the equations (1), (2), (3), (4).

Hence the equations are consistent.

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{3}(a + b + c - 2d)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{3}(a + b - 2c + d)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$+\frac{1}{3}(a - 2b + c + d)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \frac{1}{3}(-2a + b + c + d)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The given matrices generate $(M_{2\times 2}R)$.

**Example:7**

Show that the vectors (1,1,0), (1,0,1) and (0,1,1) generates $R^3$.

**Solution:**

Let $(a,b,c) \in R^3$, $\alpha_1, \alpha_2, \alpha_3 \in R$

$$(a,b,c) = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,1,1)$$
$$(a,b,c) = (\alpha_1 + \alpha_2,\ \alpha_1 + \alpha_3,\ \alpha_2 + \alpha_3)$$
$$\Rightarrow \alpha_1 + \alpha_2 = a,\quad \alpha_1 + \alpha_3 = b,\ \alpha_2 + \alpha_3 = c$$

We write the above system of equations in the matrix form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

By changing into the row echelon form,



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$$A = \begin{bmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix}$$

$$□ \begin{bmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 2 & -a+b+c \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$□ \begin{bmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 2 & -a+b+c \end{bmatrix} \quad R_3 \to R_3 + R_2$$

Now the equations are

$$\alpha_1 + \alpha_2 = a$$

$$-\alpha_2 + \alpha_3 = b - a$$

$$2\alpha_3 = (-a + b + c) \Rightarrow \alpha_3 = \frac{1}{2}(-a + b + c)$$

Substituting $\alpha_3$ in the above equations, we get

$$\alpha_1 = \frac{1}{2}(a + b - c), \ \ \alpha_2 = \frac{1}{2}(a - b + c)$$

Which satisfies the equations (1), (2), (3).

Hence the equations are consistent.

$$(a, b, c) = \frac{1}{2}(a + b - c)(1,1,0) + \frac{1}{2}(a - b + c)(1,0,1) + \frac{1}{2}(-a + b + c)(0,1,1)$$

Therefore, any vector $(a, b, c)$ can be written as a linear combination of vectors.
Hence the given vectors generate $R^3$.

### 3.1.9 Convex Combination

A convex combination is a special type of linear combination where the coefficients (scalars) are non-negative and sum up to 1. Let's consider vectors $v_1, v_2, ..., v_n$, and scalars $a_1, a_2, ..., a_n$. A convex combination is given by: $c = a_1v_1 + a_2v_2 + ... + a_nv_n$ where $a_1, a_2, ..., a_n$ satisfy the following conditions:

$a_1 \geq 0, a_2 \geq 0, ..., a_n \geq 0$ (non-negativity condition)

$a_1 + a_2 + ... + a_n = 1$ (normalization condition)

The second condition ensures that the coefficients sum up to 1, making it a convex combination.

Let's work on some examples to better understand linear and convex combinations.

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**Example** 1:

Consider two vectors in $R^2$:

$v_1 = [2, 3]$

$v_2 = [-1, 4]$

Find the linear combination:

**Solution:**

$c = 3v_1 - 2v_2$

To obtain the linear combination, we multiply each vector by its scalar and add them together:

$c = 3[2, 3] - 2[-1, 4]$

$= [6, 9] - [-2, 8]$

$= [8, 1]$

So, the linear combination of $3v_1 - 2v_2$ is [8, 1].

**Example 2**:

Consider three vectors in $R^3$:

$v_1 = [1, 0, 2]$

$v_2 = [0, 1, -1]$

$v_3 = [2, 1, 3]$

Find a convex combination of these vectors.

**Solution:**

Let's choose the coefficients as follows:

$a_1 = 0.4, a_2 = 0.3, a_3 = 0.3$

The convex combination is given by:

$c = 0.4v_1 + 0.3v_2 + 0.3v_3$

$c = 0.4[1, 0, 2] + 0.3[0, 1, -1] + 0.3[2, 1, 3]$

$= [0.4, 0, 0.8] + [0, 0.3, -0.3] + [0.6, 0.3, 0.9]$

$= [1, 0.6, 1.4]$

So, the convex combination of $v_1$, $v_2$, and $v_3$ with coefficients 0.4, 0.3, and 0.3 is [1, 0.6, 1.4].

**Exercise:**

1) Given vectors $v_1 = [2, 3, -1]$ and $v_2 = [1, -2, 4]$, find the linear combination: $c = 2v_1 + 3v_2$.



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2) Consider three vectors in $R^4$: $v_1 = [1, 0, -1, 2]$, $v_2 = [0, 1, 1, 0]$, and $v_3 = [-1, 2, 1, 3]$. Find a convex combination of these vectors with coefficients $a_1 = 0.2$, $a_2 = 0.3$, and $a_3 = 0.5$.
3) Let $v_1 = [-1, 2, 3]$ and $v_2 = [4, -2, 0]$. Determine the values of scalars a and b such that
4) $c = av_1 + bv_2$ is the midpoint of $v_1$ and $v_2$.

**Example 1**:

Consider two vectors $v_1 = [1, 2]$ and $v_2 = [3, 1]$. Find the convex combination

$c = \lambda_1 v_1 + \lambda_2 v_2$ for $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$.

**Solution:**

$$
\begin{aligned}
c &= 0.6[1, 2] + 0.4[3, 1] \\
&= [0.6 * 1, 0.6 * 2] + [0.4 * 3, 0.4 * 1] \\
&= [0.6, 1.2] + [1.2, 0.4] \\
&= [0.6 + 1.2, 1.2 + 0.4] \\
&= [1.8, 1.6]
\end{aligned}
$$

Therefore, the convex combination c of $v_1$ and $v_2$ with $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$ is $c = [1.8, 1.6]$.

**Example 2**:

Consider three vectors $v_1 = [1, 1]$, $v_2 = [-1, 3]$, and $v_3 = [2, -2]$. Find a convex combination $c = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ for $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, and $\lambda_3 = 0.3$.

**Solution:**

$$
\begin{aligned}
c &= 0.3[1, 1] + 0.4[-1, 3] + 0.3[2, -2] \\
&= [0.3 * 1, 0.3 * 1] + [0.4 * -1, 0.4 * 3] + [0.3 * 2, 0.3 * -2] \\
&= [0.3, 0.3] + [-0.4, 1.2] + [0.6, -0.6] \\
&= [0.3 - 0.4 + 0.6, 0.3 + 1.2 - 0.6] \\
&= [0.5, 0.9]
\end{aligned}
$$

Therefore, the convex combination c of $v_1$, $v_2$, and $v_3$ with $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, and $\lambda_3 = 0.3$ is $c = [0.5, 0.9]$.

Certainly! Here are a few more example problems involving convex combinations of vectors:

**Example 3**:

Consider four vectors: $v_1 = [1, 0, 2]$, $v_2 = [-1, 1, 0]$, $v_3 = [2, 1, 1]$, and $v_4 = [0, -1, 1]$. Find a convex combination $c = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4$ such that $\lambda_1$

$= 0.2, \lambda_2 = 0.3, \lambda_3 = 0.4$, and
$\lambda_4 = 0.1$.

**Solution:**

c = 0.2[1, 0, 2] + 0.3[-1, 1, 0] + 0.4[2, 1, 1] + 0.1[0, -1, 1]

= [0.2 * 1, 0.2 * 0, 0.2 * 2] + [0.3 * -1, 0.3 * 1, 0.3 * 0] + [0.4 * 2, 0.4 * 1, 0.4 * 1] + [0.1 * 0, 0.1 * -1, 0.1 * 1]

= [0.2, 0, 0.4] + [-0.3, 0.3, 0] + [0.8, 0.4, 0.4] + [0, -0.1, 0.1]

= [0.2 - 0.3 + 0.8, 0 + 0.3 - 0.1, 0.4 + 0 - 0.1]

= [0.7, 0.2, 0.3]

Therefore, the convex combination c of $v_1$, $v_2$, $v_3$, and $v_4$ with $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $\lambda_3 = 0$.

**Example 4:**

Consider three vectors: $v_1$ = [1, 2], $v_2$ = [3, 1], and $v_3$ = [-2, 0]. Find a convex combination $c = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ such that $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, and $\lambda_3 = 0.3$.

**Solution:**

c = 0.2[1, 2] + 0.5[3, 1] + 0.3[-2, 0]
= [0.2 * 1, 0.2 * 2] + [0.5 * 3, 0.5 * 1] + [0.3 * -2, 0.3 * 0]
= [0.2, 0.4] + [1.5, 0.5] + [-0.6, 0]
= [0.2 + 1.5 - 0.6, 0.4 + 0.5]
= [1.1, 0.9]

Therefore, the convex combination c of $v_1$, $v_2$, and $v_3$ with $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, and $\lambda_3 = 0.3$ is c = [1.1, 0.9].

**Example 5:**

Consider four vectors: $v_1$ = [1, 0], $v_2$ = [0, 1], $v_3$ = [1, 1], and $v_4$ = [-1, -1]. Find a convex combination $c = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4$ such that $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\lambda_3 = 0.3$, and $\lambda_4 = 0.2$.

**Solution:**

c = 0.4[1, 0] + 0.1[0, 1] + 0.3[1, 1] + 0.2[-1, -1]
= [0.4 * 1, 0.4 * 0] + [0.1 * 0, 0.1 * 1] + [0.3 * 1, 0.3 * 1] + [0.2 * -1, 0.2 * -1]
= [0.4, 0] + [0, 0.1] + [0.3, 0.3] + [-0.2, -0.2]
= [0.4 + 0.3 - 0.2, 0 + 0.1 + 0.3 - 0.2]
= [0.5, 0.2]

Therefore, the convex combination c of $v_1$, $v_2$, $v_3$, and $v_4$ with $\lambda_1 = 0.4$, $\lambda_2 =$

0.1, $\lambda_3$ = 0.3, and $\lambda_4$ = 0.2 is c = [0.5, 0.2].

**Exercise:**

Determine whether u and v are orthogonal vectors

$$(a) u = (6,1,4), v = (2,0,-3)$$
$$(b) u = (0,0,-1), v = (1,1,1)$$
$$(c) u = (-6,0,4), v = (3,1,6)$$
$$(d) u = (2,4,-8), v = (5,3,7)$$

| Answers |
|---|
| $(a)$ orthogonal |
| (b) not orthogonal |
| (c) not orthogonal |
| (d) not orthogonal |

### 3.1.10 Self Assessment Question

1. Express the vector (1,-2,5) as a linear combination of the vectors (1,1,1), (1, 2, 3) and (2, -1, 1) in $R^3$
2. Find a unit vector orthogonal to $v_1 = (1,2,1)$ and $v_2 = (3,1,0)$ in $R^3$ with standard inner product.
3. Find k so that $u = (1,2,k,3)$ and in $R^4$ are orthogonal.
4. Let u = (-1, 1/4) and v = (4, -1/8). Then find $\|u\|$.
5. Determine whether the vectors form an orthogonal set
6. $v_1 = (-3,4,-1), v_2 = (1,2,5), v_3 = (4,-3,0)$

### 3.1.11Summary

1.Vector: A $k$-dimensional vector $y$ is an ordered collection of $k$ real numbers $y_1, y_2, \ldots$ written as $y = (y_1, y_2, \ldots, y_k)$. The numbers $y_j$ $(j = 1,2, \ldots, k)$ are called the component vector $y$.

2. Scalar multiplication: Scalar multiplication of a vector $y = (y_1, y_2, \ldots, y_k)$ and a scalar $\alpha$ is defined to be a new vector $z = (z_1, z_2, \ldots, z_k)$, written $z = \alpha y$ or $z = y\alpha$, whose components are given by $z_j = \alpha y_j$

3. Vector addition: Vector addition of two $k$-dimensional vectors $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ is defined as a new vector $z = (z_1, z_2, \ldots, z_k)$, denoted $z = x + y$, with components given by $z_j = x_j + y_j$.



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4. Vector space: Set $V$ is called a vector space over a field $F$ if the operations "addition of vectors" (deno by $+$) and scalar multiplication (denoted by .) are defined such that the following hold:

- is a binary operation, i.e. $X+Y \in V$ for all $X, Y \in V$.
- is associative, i.e. $(X+Y)+Z = X+(Y+Z)$ for all $X.Y.Z \in V$.
- is commutative, i.e. $X+Y = Y+X$ for all $X, Y \in V$.

Additive identity: There exists $O \in V$ such that $O + X = X + O = X$ for all $X \in V$.

Additive inverse: To each element $X$ of $V$ there is an element $Y$ in $V$ such

$X + Y = O$. Then $Y$ is the additive inverse of $X$.It is denoted by $-X$.

Scalar multiplication is well defined i.e. $\forall\ c \in F$ and $X \in V, \Rightarrow cX \in V$

Additive inverse: To each element $X$ of $V$ there is an element $Y$ in $V$ such that

$X + Y = O. Then$ the additive inverse of $X$.It is denoted by $-X$.

Scalar multiplication is well defined i.e. $\forall\ c \in F$ and $X \in V, \Rightarrow cX \in V$

$c(X+Y) = cX + cY\ \forall\ c \in F$ and $X, Y \in V$

$(c+d)X = cX + dX\ \forall\ c, d \in F$ and $X \in V$

$(cd)X = c(dX)\ \forall\ c, d \in F$ and $X \in V$

**6. Inner Product Space:**

An inner product on a vector space $V(F)$ is a function that assigns, to every ordered pair of vectors $u, v \in V$, a scalar in $F$ such that $\forall u, v, w \in V$ and $\forall\, \alpha, \beta \in F$ the following axioms hold.

$\overline{\langle u,v\rangle} = \langle v,u\rangle$, where the bar denotes the complex conjugation

$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$

$\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ if and only if $u = 0$.

A vector space $V(F)$ with an inner product on it is called an inner product space.

**7.Norm or length of a vector**

Let be an inner product space. For $v \in V$, we define the norm or length of $v$ by $\|v\| = \sqrt{\langle v, v\rangle}$.

The vector is called a unit vector if $\|v\| = 1$.



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**8. Orthogonal Vectors**

Let $V$ be an inner product space. The vectors $v$ and $v$ in $V$ are orthogonal (or perpendicular) if $\langle u, v\rangle = 0$.A subset $V$ of $V$ is orthogonal if any two distinct vectors in $S$ are orthogonal.

9. **Orthogonal projection:**

Given vectors u, v ∈ $R^n$, where $u \neq 0$, then the **orthogonal projection** of **v** onto **u** is $x, v \in V,\ v \neq 0$

Let V be an inner product space. Let $x, v \in V,\ v \neq 0$. Then

$$P = \frac{\langle x, v\rangle}{\langle v, v\rangle} v$$

is the **orthogonal projection** of the vector x onto the vector v.

If $v_1, v_2, \ldots, v_n$ is an orthogonal set of vectors, then

$$P = \mathrm{Proj}_u v = \frac{\langle x.v_1\rangle}{\langle v_1.v_1\rangle} v_1 + \frac{\langle x.v_2\rangle}{\langle v_2.v_2\rangle} v_2 + \ldots + \frac{\langle x.v_n\rangle}{\langle v_n.v_n\rangle} v_n$$

is the **orthogonal projection** of the vector x onto the subspace spanned by $v_1, v_2, \ldots, v_n$

**10. Orthonormal basis:**

Let $V$ be an inner product space. A subset $V$ of $V$ is orthonormal basis if it is an ordered basis that is orthonormal.

**11. Gram Schmidt Orthogonalization Proc**

Every finite dimensional inner product space has an orthonormal set as a basis.

**12. Linear combination**

Let $V$ be a vector space and $S$ be a non empty subset of $V$. A vector $v \in V$ is called a linear combination of vectors of $S$, if there exists $u_1, u_2, u_3, \ldots u_n \in S$ ; $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in F$ such that $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots \alpha_n u_n$.

**13. Linear span:**

Let $S$ be a non empty subset of the vector space $V(F)$. The set of all linear combinations of finite sets of elements of $S$ is called the linear span of $S$ and it is denoted as $L(S)$ or $span(S)$.A subset $S$ of a vector space $V$ generates (or spans) $V$ if $span(S) = V$.

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**14. Convex Combination**

A convex combination is a special type of linear combination where the coefficients (scalars) are non-negative and sum up to 1. Let's consider vectors $v_1, v_2, ..., v_n$, and scalars $a_1, a_2, ..., a_n$. A convex combination is given by: $c = a_1v_1 + a_2v_2 + ... + a_nv_n$ where $a_1, a_2, ..., a_n$ satisfy the following conditions: $a_1 \geq 0, a_2 \geq 0, ..., a_n \geq 0$ (non-negativity condition)
$a_1 + a_2 + ... + a_n = 1$ (normalization condition)

The second condition ensures that the coefficients sum up to 1, making it a convex combination.

**15. Linear Combination:**

Let $V$ be a vector space and $S$ be a non empty subset of $V$. A vector $v \epsilon V$ is called a linear combination of vectors of $S$, if there exists $u_1, u_2, u_3, ... u_n \in S$ ; $\alpha_1, \alpha_2, \alpha_3, ... \alpha_n \in F$ such that $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots \alpha_n u_n$.

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# UNIT – IV

## Lesson 4.1- Elementary Calculus – Differentiation

### Structure



### 4.1.1 Objectives

1. Remember the properties of sums, products, and quotients of functions also recognize the concept of composite functions and the chain rule
2. Explain the meaning of differentiability and continuity
3. Interpret the rules and techniques of differentiation
4. Explain the concepts of sums, products, and quotients of functions
5. Describe the concept of composite functions and the chain rule
6. Apply the concepts of inverse functions to solve problems

### 4.1.2 Introduction

The derivative of a function is a fundamental concept in calculus that represents the rate at which the function changes at a particular point. Geometrically, the derivative represents the slope of the tangent line to the



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function at that point.

Differentiability and continuity are important concepts in calculus that relate to the smoothness of a function. A function is said to be differentiable at a point if its derivative exists at that point. A function is said to be continuous at a point if it does not have any abrupt changes or "jumps" at that point.

Techniques of differentiation include the power rule, product rule, quotient rule, and chain rule. The power rule states that the derivative of x^n is nx^(n-1). The product rule is used to differentiate the product of two functions, and the quotient rule is used to differentiate the quotient of two functions. The chain rule is used to differentiate composite functions.

Composite functions are functions that are made up of two or more functions. The chain rule is used to differentiate these functions by breaking them down into smaller parts and finding the derivative of each part separately.

Inverse functions are functions that "undo" each other. The derivative of an inverse function can be found using the chain rule and the fact that the derivative of the inverse function is equal to 1 divided by the derivative of the original function.

Implicit differentiation is used to find the derivative of a function that is not written in explicit form. This involves differentiating both sides of an equation with respect to the variable of interest.

Second and higher-order derivatives provide information about the curvature of a function. The second derivative represents the rate at which the slope of the tangent line changes, and can be used to determine whether a function is concave or convex.

Concavity and convexity of functions are important concepts in calculus that relate to the shape of a function. A function is said to be concave if its second derivative is negative, and convex if its second derivative is positive.

Overall, a solid understanding of the derivative of a function and its various applications is essential for success in calculus and many other areas of mathematics and science

**4.1.3 Derivatives of Polynomials and Exponential Functions**

1. Derivative of a Constant Function



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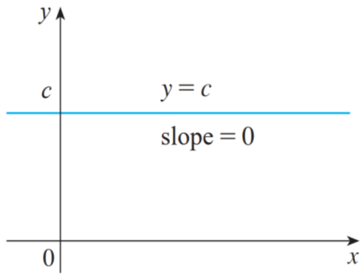
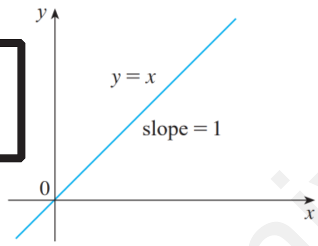
$$\boxed{\frac{d}{dx}(c) = 0}$$




2. The Power Rule

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \text{ , for any real value of.}}$$

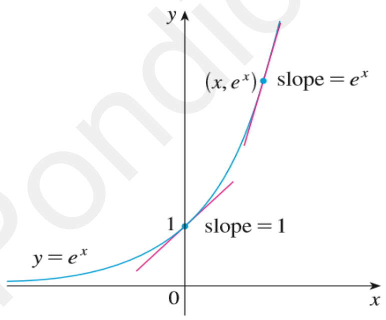



3. Derivative of the natural exponential function Derivative of a natural exponential function is again natural exponential function.

$$\frac{d}{dx}(e^x) = e^x$$

In general, $\frac{d}{dx}(e^{ax}) = ae^x$

The following graph shows the derivative of a natural exponential function.




The slope at the point (0,1) is 1. In general, the slope at the point $(x, e^x)$ is $e^x$.

Remark:

$$\boxed{\frac{d}{dx}(b^x) = b^x \log b, \quad \text{where b is a positive real number.}}$$

#### 4.1.4 Differentiability and Continuity

Differentiability and continuity are fundamental concepts in calculus and mathematical analysis. They describe the behaviour of functions and

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provide important properties for understanding their behavior.

**Continuity**

Continuity is a property of a function that describes how it behaves without any abrupt changes or jumps. A function f(x) is said to be continuous at a point x = a if three conditions are met:

a. The function is defined at x = a.

b. The limit of the function as x approaches a exists.

c. The limit of the function as x approaches a is equal to the value of the function at x = a.

If a function satisfies these conditions for all points in its domain, it is called a continuous function.

**Differentiability:**

Differentiability is a stronger condition than continuity. A function f(x) is said to be differentiable at a point x = a if the derivative of the function exists at that point. Geometrically, this means that the function has a well-defined tangent line at x = a.

The derivative of a function f(x) at a point x = a is denoted by f'(a) or $\left.\frac{dy}{dx}\right|_{x=a}$ It represents the rate of change of the function at that point.

A function is differentiable if it is differentiable at every point in its domain. If a function is differentiable, it must also be continuous, but the converse is not always true. There are functions that are continuous but not differentiable at certain points.

### 4.1.5 Relationship Between Differentiability and Continuity

If a function is differentiable at a point, it must be continuous at that point. This implies that differentiability is a stronger condition than continuity. However, continuity does not guarantee differentiability. For a function to be differentiable, it requires the existence of a well-defined tangent line, which may not be possible if the function has sharp corners, cusps, or vertical tangents.

#### (a) Differentiability Implies Continuity Theorem

The differentiability implies continuity theorem states that if a function f(x) is differentiable at a point x=a, then f(x) must be continuous at that point. This theorem provides a useful criterion for determining continuity based on



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differentiability.

In summary, continuity describes the absence of abrupt changes in a function, while differentiability extends the notion of continuity to include well-defined tangent lines. Both concepts play crucial roles in calculus and analysis, providing insights into the behavior of functions and enabling the development of powerful mathematical technique.

**Definition:**

A function $f$ is said to be differentiable in the closed interval [a,b] if it is differentiable on the open interval (a,b) and at the end points a and b,

$$f^{'}(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, h > 0$$

$$f^{'}(b) = \lim_{\Delta x \to 0} \frac{f(b+\Delta x) - f(b)}{\Delta x} = \lim_{h \to 0} \frac{f(b-h) - f(b)}{-h}, h > 0$$

If $f$ is differentiable at $x = x_0$ then $f^{'}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ where $x = x_0 + \Delta x$ and $\Delta x \to 0$ is equivalent to $x \to x_0$.

As a matter of convenience, if we let $h = \Delta x$, then

$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

**Example 1: Test the differentiability of the function** $f(x) = |x-2|$ at x=2.

**Solution:** We know that this function is continuous at x=2.

$$f^{'}(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^-} \frac{|x-2| - 0}{x-2}$$

$$= \lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)}{(x-2)} = -1$$

But

$$f^{'}(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^+} \frac{|x-2| - 0}{x-2}$$

$$= \lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{(x-2)}{(x-2)} = -1$$

Since the one-sided derivatives $f^{'}(2^-)$ and $f^{'}(2^+)$ are not equal, $f^{'}(2)$ does not exist. That is , $f$ is not differential at x=2. At all other points, the function is differentiable.

If $x_0 \neq 2$ is any other point

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$$f'(x_0) = \lim_{x \to x_0} \frac{|x - x_0|}{x - x_0} = \begin{cases} 1 \, if \; x > x_0 \\ -1 \, if \; x < x_0 \end{cases}$$

$$\text{Thus } f'(2) = \begin{cases} 1 \, if \; x > 2 \\ -1 \, if \; x < 2 \end{cases}$$

The fact that $f'(2)$ does not exist.

**Example 2:** Examine the differentiability of $f(x) = x^{\frac{1}{3}}$ at $x=0$.

**Solution:**

Let $f(x) = x^{\frac{1}{3}}$. Clearly, there is no hole (or break) in the graph of this function z And hence it is continuous at all points of its domain.
Let us check whether $f'(0)$ exists.

Now,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0}{x}$$

$$= \lim_{x \to 0} x^{\frac{-2}{3}} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} \to \infty$$

Therefore, the function is not differentiable at x=0. So, f is not differentiable at x=0.

**Note :** If a function is continuous at a point, then it is not necessary that the function is differentiable at that point.

**Exercise:**

Examine the differentiability off (x) at x=1

$$(i)\, f(x) = |x - 1| \quad (ii)\, f(x) = \sqrt{1 - x^2} \quad (iii)\, f(x) = \begin{cases} x, x \le 1 \\ x^2, x > 1 \end{cases}$$

Answer:

$(i)\, f'(1^-) = -1, f'(1^+) = 1,$ not differentiable

$(ii)\, f'(x) \to -\infty \; as \; x \to 1^-$, not differentiable

$(iii)\, f'(1^-) = 1, f'(1^+) = 2,$ not differentiable

## 4.1.6 Rules of Differentiation

1. The constant multiple rule:

$\frac{d}{dx}(c\, f(x)) = c \frac{d}{dx}(f(x))$, where $f$ is a differentiable function and c is a constant.



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Notes

2. The sum rule:

$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$ , where $f$ and g are differentiable functions.

The derivative of the sum of two (or more) differentiable functions is equal to the sum of their derivatives

3. The difference rule:

$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$ , where $f$ and g are differentiable functions.

derivative of the difference of two (or more) differentiable functions is equal to the difference of their derivatives.

4. The product rule:

$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$ , where $f$ and g are differentiable functions.

The above formula gives the rule to find the derivative of product of two functions. The same formula can be extended to find the derivative of product of three functions.

$$\frac{d}{dx}[f(x)\,g(x)\,h(x)] = f(x)\,g(x)\frac{d}{dx}[h(x)] + f(x)\,h(x)\frac{d}{dx}[g(x)] + g(x)\,h(x)\frac{d}{dx}[f(x)]$$

5. The quotient rule:

The above formula gives the rule to find the derivative of quotient of two functions.

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$ $if\ g(x) \neq 0$, where $f$ and g are differentiable functions.

6. The chain rule: Let $y = f(u)$ be a function of $u$ and in turn let $u = g(x)$ be a function of $x$ so that $y = f(g(x)) = (fog)(x)$. Then

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

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**Differentiation Formulae**

$\frac{d}{dx}(c) = 0$
$\frac{d}{dx}(x^n) = nx^{n-1}$ , for any real value of $n$.
$\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(e^{ax}) = ae^x$
$\frac{d}{dx}(b^x) = b^x logb$, where $b$ is a positive real number.
$\frac{d}{dx}(\log_a x) = \frac{1}{x log_e a}$, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -cosec^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\operatorname{cosec} x) = -cosec\, x \cot x$
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Example 1: If $f(x) = x^3$, Then $f'(x) = 3x^2$.

Example 2: If $f(x) = x^4$ Then $f'(x) = 4x^3$.

Example 3: If $f(x) = x^5$ Then $f'(x) = 5x^4$.

Example 4: Differentiate the following functions.

a) $f(x) = \frac{-1}{x^2}$ b) $f(x) = x^{1000}$

**Solution:**

Given $f(x) = -x^{-2}$. Then by power rule of differentiation, we have

$$f'(x) = -(-2)x^{-2-1} = 2x^{-3} = \frac{2}{x^3}.$$

If $f(x) = x^{1000}$ then $f'(x) = 1000x^{1000-1}$=$1000x^{999}$.

**Example 5:** Find the derivative of the function $f(x) = ax^3 + cx$.

Solution: Given $f(x) = ax^3 + cx$.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(ax^3 + cx)$$

$$= \frac{d}{dx}(ax^3) + \frac{d}{dx}(cx) \quad \text{(using sum rule)}$$

The product rule:

$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$ , where $f$ and g are differentiable functions.

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**Example:1** Differentiate the function

(a) $f(x) = (x-2)(2x+3)$

Given $f(x) = (x-2)(2x+3) = g(x)h(x)$ ,

where $g(x) = (x-2)$ and $h(x) = (2x+3)$.

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$
$$= (x-2)(2) + (2x+3)(1)$$
$$=2x—4+2x+3$$
$$= 4x - 1.$$

**Example:2** Find $f'(x)$, if $f(x) = e^x\left(x + x\sqrt{x}\right)$.

**Solution:** Given $f(x) = e^x\left(x + x\sqrt{x}\right)$

$$\frac{d}{dx}(f(x)) = f'(x) = e^x\left[1 + x.\frac{1}{2}x^{-\frac{1}{2}} + \sqrt{x}.1\right] + \left(x + x\sqrt{x}\right)e^x$$

$$\frac{d}{dx}(f(x)) = f'(x) = e^x\left[1 + x.\frac{1}{2\sqrt{x}} + \sqrt{x}\right] + \left(x + x\sqrt{x}\right)e^x$$

$$= e^x\left[1 + \sqrt{x}.\frac{1}{2} + \sqrt{x}\right] + \left(x + x\sqrt{x}\right)e^x$$

$$= e^x\left[1 + \frac{3\sqrt{x}}{2}\right] + \left(x + x\sqrt{x}\right)e^x$$

$$= e^x\left[1 + \frac{3\sqrt{x}}{2} + x + x\sqrt{x}\right]$$

**Example:3** Find the derivative of $y(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)\left(x + 5x^3\right)$.

**Solution:** Given $y(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)\left(x + 5x^3\right)$

$$y'(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)\left(1 + 15x^2\right) + \left(x + 5x^3\right)\left(\frac{-2}{x^3} + \frac{12}{x^5}\right)$$

$$= \frac{1}{x^2} - \frac{3}{x^4} + 15 - \frac{45}{x^2} - \frac{2}{x^2} - 10 + \frac{12}{x^4} + \frac{60}{x^2}$$

$$= \frac{9}{x^4} + \frac{14}{x^2} + 5$$

**Example: 4** Find the derivative of $\left(\frac{1}{x^2} + x^2\right)\tan x$.

**Solution:** Let $y(x) = \left(\frac{1}{x^2} + x^2\right)\tan x$

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$$y'(x) = \left(\frac{1}{x^2} + x^2\right)\sec^2 x + \tan x\left(\frac{-2}{x^3} + 2x\right)$$

**Example:5** Find the derivative of f(x) if $f(x) = (a+bx)\sqrt{x}$

**Solution**: The given function is

$$f(x) = (a+bx)\sqrt{x}$$

Differentiating using the product rule

$$f'(x) = \frac{d}{dx}\left[(a+bx)\sqrt{x}\right]$$

$$= (a+bx)\frac{1}{2\sqrt{x}} + \sqrt{x}b$$

$$= \frac{(a+bx)+2bx}{2\sqrt{x}}$$

$$= \frac{a+3bx}{2\sqrt{x}}$$

**Exercise problems**

Find the first and second derivatives of the functions.

$(a)$ $x^4 - 4x^3 + 16x$ (b) $e^x - x$ (c) $xe^x$ (d) $x^2 \sin x$

**Answers:**

$4x^3 - 12x^2 + 16,\ \ 12x^2 - 24x$ (b) $e^x - 1,\ e^x$ (c) $xe^x + e^x, xe^x + 2e^x$

(d) $x^2 \cos x + 2x\sin x$

### 4.1.7 Composite Functions and the Chain Rule

The chain rule is an important concept in calculus that allows us to find the derivative of composite functions. A composite function is a function within another function, and the chain rule provides a method for finding its derivative.

The chain rule states that the derivative of a composite function f(g(x)) is equal to the product of the derivative of the outer function f'(g(x)) with the derivative of the inner function g'(x). In other words:

(f(g(x)))' = f'(g(x)) * g'(x)

If a composite function $r(x)$ is defined as



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$$r(x) = (m \circ n \circ p)(x) = m\{n[p(x)]\}$$
$$\text{then } r'(x) = m'\{n[p(x)]\} \cdot n'[p(x)] \cdot p'(x)$$

Here, three functions— $m$, $n$, and $p$—make up the composition function $r$; hence, you have to consider the derivatives $m'$, $n'$, and $p'$ in differentiating $r(x)$. A technique that is sometimes suggested for differentiating composite functions is to work from the "outside to the inside" functions to establish a sequence for each of the derivatives that must be taken.

$$\text{Example 1: Find } f'(x) \text{ if } f(x) = (3x^2 + 5x - 2)^8$$
$$f'(x) = 8(3x^2 + 5x - 2)^7 .(6x + 5)$$
$$8(6x + 5).(3x^2 + 5x - 2)^7$$

$$\text{Example 2: Find } f'(x) \text{ if } f(x) = \tan(\sec x)$$
$$f'(x) = \sec^2(\sec x).\sec x \tan x$$
$$\sec^2(\sec x).\sec x \tan x$$

$$\text{Example 3: Find } \frac{dy}{dx} \text{ if } y = \sin^3(3x-1)$$
$$\frac{dy}{dx} = 3\sin^3(3x-1).\cos(3x-1).(3)$$
$$= 9\cos(3x-1)\sin^2(3x-1)$$

$$\text{Example 4: Find } f'(2) \text{ if } f(x) = \sqrt{5x^2 + 3x - 1}$$
$$f(x) = (5x^2 + 3x - 1)^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}(5x^2 + 3x - 1)^{-\frac{1}{2}}(10x + 3)$$
$$= \frac{10x + 3}{2\sqrt{5x^2 + 3x - 1}}$$
$$f'(2) = \frac{10.2 + 3}{2\sqrt{5(x)^2 + 3x - 1}}$$
$$= \frac{23}{2\sqrt{25}}$$
$$= \frac{23}{10}$$

Example 5: Find the slope of the tangent line to a curve $y=(x^2-3)^5$ at the point(-1,-32)

Because the slope of the tangent line to a curve is the derivative, you find that



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$$y' = 5(x^2-3)^4(2x)$$
$$= 10x(x^2-3)^4$$
$$\text{hence,at}(-1,-32)\, y' = 10(-1)[(-1)^2-3]^4$$
$$= (-10)(-2)^4$$
$$= -160$$

Which represents the slope of the tangent line at the point (-1,-32)

### 4.1.8 Derivative of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

**Example:1** Find the derivative of the following functions:

(a) $\mathrm{f(x)} = \mathrm{x}\tan^{-1}(\sqrt{x})$

(b) $y = (1+x^2)\tan^{-1}x$

**Solution:**

(a) Given, $\mathrm{f(x)} = \mathrm{x}\tan^{-1}(\sqrt{x})$

$$\Rightarrow \frac{dy}{dx} = x\frac{1}{1+\sqrt{x}^2}\cdot\frac{1}{2\sqrt{x}} + \tan^{-1}\sqrt{x}\,.\,1 = \frac{\sqrt{x}}{2+2x} + \tan^{-1}\sqrt{x}$$

(b) $y = (1+x^2)\tan^{-1}x$

$$\frac{dy}{dx} = (1+x^2)\frac{1}{1+x^2} + (\tan^{-1}x)(2x) = 1 + 2x\tan^{-1}x$$

The Quotient rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ $if\ g(x) \neq 0$, where $f$ and g are differentiable functions.

**Example:1** Find the derivative of $y = \frac{\mathrm{x}-2}{2\mathrm{x}+1}$



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**Solution:**

(b) Given $= \frac{x-2}{2x+1}$ . Then,

$$\frac{dy}{dx} = \frac{(2x+1)(1-0)-(x-2)(2+0)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2x+1-2x+4}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{5}{(2x+1)^2}$$

**Example:2** Find the derivative of $f(x) = \frac{1-xe^x}{x+e^x}$ .

**Solution:** Given $f(x) = \frac{1-xe^x}{x+e^x}$

$$f'(x) = \frac{(x+e^x)\left[0-(xe^x+e^x.1)\right]-(1-xe^x)(1+e^x)}{(x+e^x)^2}$$

$$= \frac{-x^2e^x \cancel{-xe^{2x}} \cancel{-xe^x} - e^{2x} - 1 - e^x \cancel{+xe^x} \cancel{+xe^{2x}}}{(x+e^x)^2}$$

$$= \frac{-(x^2e^x+e^{2x}+e^x+1)}{(x+e^x)^2}$$

**Example:3** Find the derivative of $f(x) = \frac{x^3-2x\sqrt{x}}{x}$ .

**Solution:** Given $f(x) = \frac{x^3-2x\sqrt{x}}{x} = \frac{x^3-2x^{\frac{3}{2}}}{x}$

$$f'(x) = \frac{x\left(3x^2-2.\frac{3}{2}x^{1/2}\right)-\left(x^3-2x^{3/2}\right).1}{x^2}$$

$$= \frac{3x^3-3x^{3/2}-x^3+2x^{3/2}}{x^2}$$

$$= \frac{2x^3-x^{3/2}}{x^2}$$

$$\therefore f'(x) = 2x - x^{-1/2}$$

**Example:4** Find the derivative of $f(x) = \frac{\sec x}{1+\tan x}$ .

**Solution:** Given $f(x) = \frac{\sec x}{1+\tan x}$

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$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x\left(\sec^2 x\right)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x\left(\tan^2 x - \sec^2 x\right)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x(-1)}{(1+\tan x)^2} \qquad \left(\because 1+\tan^2 x = \sec^2 x\right)$$

$$f'(x) = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

**Example: 5** Find the derivative of $y = \frac{x^2+4x+3}{\sqrt{x}}$

**Solution:**

Given $y = \frac{x^2+4x+3}{\sqrt{x}}$

$y = \frac{x^2+4x+3}{x^{\frac{1}{2}}} = \frac{g(x)}{h(x)}$, where $g(x) = x^2 + 4x + 3$ and .

$h(x) = x^{\frac{1}{2}}$

$$y' = \frac{dy}{dx} = \frac{h(x)g'(x) - g(x)h'(x)}{(g(x))^2} = \frac{x^{\frac{1}{2}}(2x+4) - (x^2+4x+3)\frac{1}{2}x^{\frac{1}{2}-1}}{\left(x^{\frac{1}{2}}\right)^2}$$

$$= \frac{x^{\frac{1}{2}}(2x+4) - (x^2+4x+3)\frac{1}{2}x^{-\frac{1}{2}}}{x}.$$

**Example: 6** Find the derivative of $\frac{x+1}{\sqrt{x-2}}$

**Solution:**

Given $y = \left(\frac{x+1}{\sqrt{x-2}}\right)$

$$\frac{dy}{dx} = \frac{\sqrt{x-2}(1+0) - (x+1)\frac{1}{2\sqrt{x-2}}}{\left(\sqrt{x-2}\right)^2} =. \frac{\frac{2\sqrt{x-2}\sqrt{x-2}}{2\sqrt{x-2}} - \frac{(x+1)}{2\sqrt{x-2}}}{\left(\sqrt{x-2}\right)^2}$$

$$= \frac{1}{2\sqrt{x-2}} \cdot \frac{2(x-2) - (x+1)}{\left(\sqrt{x-2}\right)^2}$$

**Example: 7** Find the derivative of $f(x) = \frac{b + a\cos x}{a + b\cos x}$.



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**Solution:** Given $f(x) = \frac{b+a\cos x}{a+b\cos x}$.

Differentiating with respect to '$x$', we have

$$f'(x) = \left[\frac{(a+b\cos x)(0-a\sin x)-(b+a\cos x)(0-b\sin x)}{(a+b\cos x)^2}\right]$$

$$= \left[\frac{-a^2\sin x - ab\sin x\cos x + b^2\sin x + ab\sin x\cos x}{(a+b\cos x)^2}\right]$$

$$= \left[\frac{-a^2\sin x + b^2\sin x}{(a+b\cos x)^2}\right]$$

**Example: 8** If $y = \frac{x}{\sqrt{1-x^2}}$, find $\frac{dy}{dx}$.

**Solution**: Given $y = \frac{x}{\sqrt{1-x^2}}$,

Differentiating with respect to '$x$' on both sides

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{\sqrt{1-x^2}.1 - x\left(\frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)\right)}{\left(\sqrt{1-x^2}\right)^2}\right]$$

$$= \left[\frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{(1-x^2)}\right]$$

$$= \left[\frac{1-\cancel{x^2}+\cancel{x^2}}{\sqrt{1-x^2}\,(1-x^2)}\right]$$

$$\therefore \frac{dy}{dx} = \left[\frac{1}{(1-x^2)\sqrt{1-x^2}}\right]$$

**Exercise Problems**

1. Find $f'(x)$, if $f(x) = \frac{1-x}{2+x}$.

2. Find $\frac{dy}{dx}$, if $y = x^2e^{2x}(x^2+1)^4$



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3. If $y = (\sec x + \tan x)(\sec x - \tan x)$, find $\frac{dy}{dx}$.

4. If, $y = \frac{\cot x}{1 + \cot x}$, find $\frac{dy}{dx}$.

5. If $y = \sin x \tan x$, find $\frac{dy}{dx}$.

Answers:

1. $-\frac{3}{(2+x)^2}$ 2. $2xe^{2x}(x^2+1)^4 + 4x^2e^{2x}(x^2+1)^4 + 8x^3e^{2x}(x^2+1)^3$

3. $\frac{dy}{dx} = 0$ 4. $\frac{-\operatorname{cosec}^2 x}{(1+\cot x)^2}$ 5. $\sin x \sec^2 x + \sin x$

### 4.1.9 Implicit Differentiation

Implicit differentiation is a powerful tool in calculus that is used to find the derivative of a function that is not written in explicit form. This technique involves differentiating both sides of an equation with respect to the variable of interest, typically x, and then solving for $\frac{dy}{dx}$, the derivative of y with respect to x.

To apply implicit differentiation, we assume that y is a function of x, and we differentiate both sides of the equation with respect to x using the chain rule whenever we encounter a term involving y. For example, if we have the equation $x^2 + y^2 = 25$, we will differentiate both sides with respect to x as follows:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Thus, we have found the derivative of y with respect to x without having to solve for y explicitly. Implicit differentiation can be used to find the derivatives of more complex functions that are not easily differentiated using standard techniques. It is especially useful for finding the derivatives of functions that are defined implicitly, such as the equations of curves and surfaces in geometry.

Example 1: Find $\frac{dy}{dx}$ *if* $x^2y^3 - xy = 10$



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**Solution:**

Diffrentiating implicity with respect to x, you find that

$$2xy^3 + x^2.3y^2.\frac{dy}{dx} - 1y - x.1.\frac{dy}{dx} = 0$$

$$3x^2\, y^2 \frac{dy}{dx} - x\frac{dy}{dx} = y - 2xy^3$$

$$(3x^2\, y^2 - x)\frac{dy}{dx} = y - 2xy^3$$

$$\frac{dy}{dx} = \frac{y - 2xy^3}{3x^2\, y^2 - x}$$

Example 2:Find $y'\, if\;\; y = \sin x + \cos y$

Diffrentiating implicity with respect to x, you find that

$$1.y' = \cos x - \sin y.y'$$

$$1.y' + \sin y.y' = \cos x$$

$$y'(1 + \sin y) = \cos x$$

$$y' = \frac{\cos x}{1 + \sin y}$$

*Example* $3: \textit{Find } y'\, at\, (-1,1)\, if\;\; x^2 + 3xy + y^2 = -1$

Diffrentiating implicity with respect to x, you find that

$$2x + 3y + 3x.y' + 2y.y' = 0$$

$$x.y' + 2y.y' = -2x - 3y$$

$$y'(3x + 2y) = -2x - 3y$$

$$y' = \frac{-2x - 3y}{(3x + 2y)}$$

At the point(-1,1),

$$y' = \frac{(-2)(-1) - 3(1)}{3(-1) + 2(1)}$$

$$= \frac{-1}{-1}$$

$$= -1$$

Example 4:Find the slope of the tangent line to the curve $x^2 + y^2 = 25$ at the point (3,-4) Because the slope of the tangent line to a curve is the derivative , deiifferntiate implicity with respect to x,which yeilds

$$2x + 2y.y' = 0$$

$$2y.y' = -2x$$

$$y' = \frac{-2x}{2y} \qquad = \frac{-x}{y}$$

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Hence , at (3,-4), $y' = \frac{-3}{-4} = \frac{3}{4}$ , and the tangent line has slope $\frac{3}{4}$ at the point(3,-4)

Example 5: Find $\frac{dy}{dx}$ by implicit differentiation: $3x + 2y = \cos y$.

**Solution** :

The given equation is,

$3x + 2y = \cos y$

Differentiating both sides with respect to x :

$$\frac{3d}{dx}(x) + \frac{2d}{dx}(y) = \frac{d}{dx}(\cos y)$$

$$(1) + 2\left(\frac{dy}{sx}\right) = -\sin y \frac{dy}{dx}$$

$$-2\left(\frac{dy}{dx}\right) - \sin y \frac{dy}{dx} = 3$$

$$-\frac{dy}{dx}(2 + \sin y) = 3$$

$$\frac{dy}{dx} = -\frac{3}{(2 + \sin y)}$$

**Answer** : The implicit derivative, $\frac{dy}{dx} = -\frac{3}{(2 + \sin y)}$.

**EXCERSISE:**

1) Find the implicit derivative y' if the function is defined as $x + ay^2 = \sin y$, where 'a' is a constant.

2) Find the second implicit derivative if $x^2 + y^2 = 4$.

3) Find the implicit derivative $\frac{dy}{dx}$ when $x^2 + 3xy + y^2 = -1$.

4)differentiate each of the following with respect to x and find $\frac{dy}{dx}$

$a) \sin y + x^2 + 4y = \cos x$

$b) 3xy^2 + \cos y^2 = 2x^3 + 5$

$c) 5x^2 - x^3 \sin y + 5xy = 10$

$d) x - \cos x^2 + \frac{y^2}{x} + 3x^5 = 4x^3$

$e) \tan 5y - y \sin x + 3xy^2 = 9$

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**ANSWERS:**

1. y' = 1/(cos y - 2ay).
2. The second implicit derivative is, $y'' = [-y^2 - x^2]/y^3$.
3. The implicit derivative dy/dx when $x^2 + 3xy + y^2 = -1$ is $-(2x+3y)/(3x+2y)$
4. $$a)\frac{dy}{dx} = \frac{-\sin x - 2x}{4 + \cos y}$$
$$b)\frac{dy}{dx} = \frac{6x^2 - 3y^2}{6xy - 2y\sin y^2}$$
$$c)\frac{dy}{dx} = \frac{10x - 3x^2 \sin x + 5y}{x^3 \cos y - 5x}$$
$$d)\frac{dy}{dx} = \frac{12x^4 - 15x^6 + y^2 - 2x^3 \sin x^2 - x^2}{2xy}$$
$$e)\frac{dy}{dx} = \frac{y\cos x - 3y^2}{5\sec^2 5y - \sin x + 6xy}$$

### 4.1.10 Higher Order Derivatives

A higher order derivative of a function is a derivative that has been taken multiple times. Specifically, the nth derivative of a function f(x) is obtained by differentiating f(x) n times with respect to x. The nth derivative is denoted by $f^{(n)}(x)$ *or* $y^{(n)}$, where the superscript "n" indicates the order of the derivative.

For example, suppose f(x) $= x^3 - 2x$. The first derivative of f(x) is $f'(x) = 3x^2 - 2$ , and the second derivative is f''(x) = 6x. The third derivative is f'''(x) = 6, and the fourth derivative is f''''(x) = 0.

Higher order derivatives can provide important information about the behavior of a function. For example, the second derivative can tell us about the concavity of a function and whether it is increasing or decreasing, while the third derivative can provide information about inflection points.

However, as the order of the derivative increases, the computation and interpretation of the derivative can become increasingly difficult, especially for more complex functions. Additionally, higher order derivatives may not be needed or relevant for many applications.

**Example 1**: Find the first, second and third derivatives of

$$f(x) = 5x^4 - 3x^3 + 7x^2 - 9x + 2$$

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$$f'(x) = 20x^3 - 9x^2 + 14x - 9$$
$$f''(x) = f^{(2)}(x) = 60x^2 + 18x - 14$$
$$f'''(x) = f^{(3)}(x) = 120x - 18$$

**Example 2**: Find the first, second and third derivatives of $y = \sin^2 x$

$$y' = 2\sin x \cos x$$
$$y'' = 2\cos x \cos x + 2\sin x(-\sin x)$$
$$= 2\cos x^2 - 2\sin^2 x$$
$$y''' = 2.2\cos x(-\sin x) - 2.2\sin x \cos x$$
$$= -4\sin x \cos x - 4\sin x cox$$
$$= -8\sin x \cos x$$

Example 3:Find $f^{(3)}(4)\, if\ f(x) = \sqrt{x}$

**Solution:**

Because $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f''(x) = \frac{1}{4}x^{-\frac{3}{2}}$$
$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

hence, $f'''(4) = \frac{3}{8}(4)^{-\frac{5}{2}}$

$$= \frac{3}{8}\left(\frac{1}{32}\right)$$
$$= \frac{3}{256}$$

### 4.1.11 Concavity and Convexity of the Function

Concave on an interval if, for any two points $x_1$ and $x_2$ within that interval and any value of t between 0 and 1, the following inequality holds:

$$f\left(tx_1 + (1-t)x_2\right) \geq tf\left(x_1\right) + (1-t)f\left(x_2\right).$$

In other words, the line segment connecting any two points on the graph of the function lies below the graph of the function itself.

Convex on an interval if, for any two points $x_1$ and $x_2$ within that interval and any value of t between 0 and 1, the following inequality holds:

$$f\left(tx_1 + (1-t)x_2\right) \leq tf\left(x_1\right) + (1-t)f\left(x_2\right).$$

In other words, the line segment connecting any two points on the graph of the function lies above the graph of the function itself.

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Find the Intervals of Concavity and Convexity for the Functions

**Example 1**: $f(x) = 3x - x^3$

**Solution:**

$$f(x) = 3x - x^3$$
$$f''(x) = -6x$$
$$-6x = 0$$
$$x = 0$$
$$convex: (-\infty, 0)$$
$$concave: (0, \infty)$$

**Example 2**: $f(x) = x^4 - 2x^2 - 8$

**Solution:**

$$f(x) = x^4 - 2x^2 - 8$$
$$f''(x) = 12x^2 - 4$$
$$12x^2 - 4 = 0$$
$$x = \pm\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$concave: (-\infty, -\frac{\sqrt{3}}{3}) \cup (\frac{\sqrt{3}}{3}, \infty)$$
$$convex: (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$$

**Example 3:** $f(x) = \dfrac{x^3}{(x-1)^2}$.

**Solution:**

$$f(x) = \frac{x^3}{(x-1)^2}$$
$$f'(x) = \frac{x^3 - 3x^2}{(x-1)^3}$$
$$f''(x) = \frac{6x}{(x-1)^4}$$
$$= \frac{6x}{(x-1)^4} = 0$$

$$convex: (0,1) \cup (1, \infty)$$
$$concave: (-\infty, 0)$$

**Example 4**: $f(x) = \dfrac{x^4 + 1}{x^2}$.

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**Solution:**

$$f(x)=\frac{x^4+1}{x^2}$$

$$f'(x)=\frac{2(x^4+1)}{x^3}$$

$$f''(x)=\frac{2(x^4+3)}{x^4}$$

$$2(x^4+3)=0$$

$$x=\sqrt[4]{-3}$$

$convex:(-\infty,0)\cup(0,\infty)$

**Example 5**: $f(x)=\dfrac{x^2}{2-x}$

**Solution:**

$$f(x)=\frac{x^2}{2-x}$$

$$f'(x)=\frac{4x-x^2}{(2-x)^2}$$

$$f''(x)=\frac{8}{(2-x)^3}$$

$$\frac{8}{(2-x)^3}=0$$

No solution

$convex:(-\infty,2)$

$concave:(2,\infty)$

**Example 6:** $f(x)=\dfrac{x}{1+x^2}$.

**Solution:**

$$f(x)=\frac{x}{1+x^2}$$

$$f'(x)=\frac{1-x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2x^3-6x}{(1+x^2)^3}$$

$$\frac{2x^3-6x}{(1+x^2)^3}=0$$

$$x=0$$

$$x=\pm\sqrt{3}$$

$convex:(-\sqrt{3},0)\cup(\sqrt{3},\infty)$

$concave:(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$

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**EXCERCISE:**

Find the Intervals of Concavity and Convexity for the Functions

$$1)\, f(x) = x + \sqrt{x}$$

$$2)\, f(x) = e^{-x^2}$$

$$3)\, f(x) = e^{\frac{1}{x}}$$

$$4)\, f(x) = (x-1)e^{-x}$$

$$5)\, f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

**SOLUTION:**

1)Concave:$(0,\infty)$

2)Convex:$(-\infty,-\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2},\infty)$

Concave:$(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

3)Convex:$(-\frac{1}{2},0) \cup (0,\infty)$

Concave:$(-\infty,-\frac{1}{2})$

4)Convex:$(3,\infty)$

Concave:$(-\infty,3)$

5)Convex:$(-\infty,-1) \cup (1,\infty)$

Concave:$(-1,1)$

**4.1.12 Self Assessment Questions**

1.Differentiate $y = \left(x^3 - 1\right)^{100}$.

$\left[\text{Ans:} 300\text{x}^2 (x^3-1)^{99}\right]$

2.Find the derivative $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$.

[Ans: 1]

3.Differentiate $(2x+1)^5\left(x^3 - x + 1\right)^4$.

[Ans: $2(2x+1)^4(x^3-x+1)^3\left(17x^3+6x^2-9x+3\right)$]

4.Find if $x^4 + x^2y^3 - y^5 = 2x + 1$.

$x^4 + x^2y^3 - y$

$$\left[\text{Ans:} \frac{dy}{dx} = \frac{2-4x^3-2xy^3}{3x^2y^2-5y^4}\right]$$

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5. Find the Intervals of Concavity and Convexity for the Functions

$$f(x) = \frac{\ln x}{x}$$

$$\left[\text{Ans:Convex:}(e^{\frac{3}{2}}, \infty) \ \& \ \text{Concave:}(-0, e^{\frac{3}{2}})\right]$$

### 4.1.13 Summary

1.Derivatives of Polynomials and Exponential Functions

$$\frac{d}{dx}(c) = 0$$

$\frac{d}{dx}(x^n) = nx^{n-1}$ , for any real value of $n$.

$\frac{d}{dx}(e^x) = e^x$ , $\frac{d}{dx}(e^{ax}) = ae^x$

$\frac{d}{dx}(b^x) = b^x logb,$ where $b$ is a positive real number.

$\frac{d}{dx}(\log_a x) = \frac{1}{xlog_e a},$ $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -cosec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\text{cosec}\, x) = -cosec\ x \cot x$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \text{v}\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\text{v}\frac{du}{dx} - u\frac{dv}{dx}}{\text{v}^2}$$

$$\frac{d}{dx}(f(g(x))) = f'\big(g(x)\big)g'(x)$$

2.The product rule: $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}\big(g(x)\big) + g(x)\frac{d}{dx}(f(x))$ ,

where $f$ and g are differentiable functions.

3.Derivative of inverse trigonometric functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

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4.The Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$ $if\ g(x) \neq 0,$

where $f$ and g are differentiable functions.

5.Concavity and convexity of the function

- Concave on an interval if, for any two points $x_1$ and $x_2$ within that interval and any value of t between 0 and 1, the following inequality holds:

$$f\left(tx_1 + (1-t)x_2\right) \geq tf(x_1) + (1-t)f(x_2).$$

In other words, the line segment connecting any two points on the graph of the function lies below the graph of the function itself.

- Convex on an interval if, for any two points $x_1$ and $x_2$ within that interval and any value of t between 0 and 1, the following inequality holds:

$$f\left(tx_1 + (1-t)x_2\right) \leq tf(x_1) + (1-t)f(x_2).$$

In other words, the line segment connecting any two points on the graph of the function lies above the graph of the function itself.

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## UNIT- V

## Lesson 5.1 - Introduction to Integration

### Structure



### 5.1.1Objectives

- Recall the definition of integration and its relation to differentiation.
- Evaluate the validity of a given mathematical statement about definite integrals.

### 5.1.2 Introduction

Integral Calculus provides powerful tools for modelling physical problems which involve continuously varying quantities. Intuitively, integration is summing up the incremental changes a quantity undergoes ultimately and tells how much the quantity changes in the long term. There is a connection between integral calculus and differential calculus. The Fundamental Theorem of Calculus relates the integral to the derivative, and we will see in this unit that it greatly simplifies the solution of many problems.

In many situations, the information about a function's derivative is not sufficient to trace back the curves. We need to find the function itself known as primitive or anti-derivative. For example, a physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water leaks from a tank wants to know the amount leaked over a certain period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the population's size future time. In each case, the problem is to find a function $F$ whose derivative is a known function $F$. If

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such a function $F$ exists, it is called an anti-derivative. That is obtained by the concept of what is known as integration.

The development of integral calculus arises out of the efforts of solving the problems of the following types:

1. The problem of finding a function whenever its derivative is given,
2. The problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of integrals namely indefinite and definite integrals, which together constitute the Integral Calculus.

**5.2 Indefinite integrals:**

An indefinite integral is an antiderivative of a function. Given a function f(x), an indefinite integral of f(x) is a function F(x) whose derivative is f(x).

The indefinite integral of a function f(x) is denoted by ∫ f(x) dx, where the symbol ∫ represents integration, f(x) is the integrand, and dx indicates the variable with respect to which the integration is being performed.

To find the indefinite integral of a function, we need to use integration rules and techniques, such as:

1. Power rule: $\int x^n dx = \frac{(x^{(n+1)})}{(n+1)} + C$, where C is the constant of integration.

2. Integration by substitution: ∫ f(g(x)) g'(x) dx = ∫ f(u) du, where u = g(x) and $\frac{dv}{dx}$ =g'(x)

3. Integration by parts: ∫ u dv = uv - ∫ v du, where u and v are functions of x and du/dx and $\frac{dv}{dx}$ are their derivatives.

4. Trigonometric substitutions: These are used to evaluate integrals that involve trigonometric functions.

5. Partial fraction decomposition: This technique is used to simplify the integration of rational functions.

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It is important to note that the indefinite integral of a function is not unique, as it can differ by a constant of integration. Therefore, we usually add the constant of integration (C) to the solution of an indefinite integral.

1 $\int cf(x)dx = c\int f(x)dx$

2 $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

3 $\int kdx = kx + C$

4 $\int x^n dx = \frac{x^{n+1}}{n+1} + C(n \neq -1)$

5 $\int e^x dx = e^x + C$

6 $\int \frac{1}{x} dx = \ln|x| + C$

7 $\int a^x dx = \frac{a^x}{\ln a} + C$

8 $\int \sin x dx = -\cos x + C$

9 $\int \cos x dx = \sin x + C$

10 $\int \sec^2 x dx = \tan x + C$

11 $\int \csc^2 x dx = -\cot x + C$

12 $\int \sec x \tan x dx = \sec x + C$

13 $\int \csc x \cot x dx = -\csc x + C$

14 $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

15 $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

16 $\int \sinh x dx = \cosh x + C$

17 $\int \cosh x dx = \sinh x + C$

18 $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

19 $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C/\log\left(x + \sqrt{x^2-1}\right) + C$

20 $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C/\log\left(x + \sqrt{x^2+1}\right) + C$

**Example 1:** Evaluate $\int \frac{1}{\sin^2 x \cos^2 x} dx.$

**Solution:** $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$$

$$= \int \sec^2 x dx + \int \csc^2 dx$$

$$= \tan x - \cot x + C.$$



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**Example 2:** Evaluate $\int \frac{\cos^2 x}{1-\sin x} dx.$

**Solution:** $\int \frac{\cos^2 x}{1-\sin x} dx = \int \frac{1-\sin^2 x}{1-\sin x} dx$

$$= \int \frac{(1+\sin x)(1-\sin x)}{1-\sin x} dx$$

$$= \int (1+\sin x)\, dx$$

$$= x - \cos x + C.$$

**Example 3:** Evaluate $\int 2^x (1+5^x)\, dx.$

**Solution:** $\int 2^x (1+5^x)\, dx = \int (2^x + 10^x) dx$

$$= \frac{2^x}{\log 2} + \frac{10^x}{\log 10} + C.$$

**Example 4: Evaluate** $\int \left( ax + \frac{b}{x^2} \right) dx.$

**Solution:** $\int \left( ax + \frac{b}{x^2} \right) dx = a\int x dx + b\int \frac{1}{x^2} dx$

$$= a\frac{x^2}{2} - \frac{b}{x} + C.$$

**Example 5: Evaluate** $\int \left( x + \frac{1}{x^2} \right)^2 dx.$

**Solution:** $\int \left( x + \frac{1}{x^2} \right)^2 dx = \int \left( x^2 + \frac{2}{x} + \frac{1}{x^4} \right) dx$

$$= \frac{x^3}{3} + 2\log x - \frac{1}{3x^3} + C.$$

**Example 6: Evaluate** $\int x^2 (1-x)^2\, dx.$

**Solution:** $\int x^2 (1-x)^2\, dx = \int x^2 (1 + x^2 - 2x)\, dx$

$$= \int (x^2 + x^4 - 2x^3) dx$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - 2\frac{x^4}{4}$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^4}{2} + C.$$



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**Example 7: Evaluate** $\int (ax+b)^4\,dx.$

**Solution:** $\int (ax+b)^4\,dx = \dfrac{(ax+b)^{4+1}}{(4+1)(a)}$

$$= \frac{(ax+b)^5}{5a} + C.$$

**Example 8: Evaluate** $\int e^{2-6x}\,dx.$

**Solution:** $\int e^{2-6x}\,dx = \dfrac{e^{2-6x}}{-6} + C.$

**Example 9: Evaluate** $\int \dfrac{2dx}{1+\cos 2x}.$

**Solution:** Let $I = \int \dfrac{2dx}{1+\cos 2x}$

$$= \int \frac{2dx}{2\cos^2 x}$$

$$= \int \sec^2 x\,dx$$

$$= \tan x + C.$$

**Example 10: Evaluate** $\int \dfrac{\cos x}{\sin^2 x}\,dx.$

**Solution:** Let $I = \int \dfrac{\cos x}{\sin x \sin x}\,dx$

$$= \int \cot x \operatorname{cosec} x\,dx$$

$$= -\operatorname{cosec} x + C\,.$$

**Example 11: Evaluate** $\int \sqrt{1+\sin 2x}\,dx.$

**Solution:** $\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}$

$$= \int \sqrt{(\sin x + \cos x)^2}\,dx$$

$$= \int (\sin x + \cos x)\,dx$$

$$= -\cos x + \sin x + C$$

**Example 12: Evaluate** $\int \dfrac{dx}{1+\sin x}$

**Solution:** $\int \dfrac{dx}{1+\sin x} = \int \dfrac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$

$$= \int \frac{(1-\sin x)\,dx}{1-\sin^2 x}$$



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$$= \int \left( \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int \sec^2 x dx - \int \tan x \sec x dx$$

$$= \tan x - \sec x + C$$

**Exercise:**

(1). $\int (4-2x)^{-3} dx$ (2). $\int \sqrt{(3-5x)} dx$ (3). $\int \left( x + \frac{1}{x} \right)^3 dx$

(4). $\int e^{5x+2} dx$ (5). $\int \frac{1}{(4x+5)^2} dx$ (6). $\int \left( x^{2/5} - x^{-3/5} \right)^2$

(7). $\int \frac{\sin^2 x}{1+\cos x} dx$ (8). $\int \cos^3 2x dx$

(9). $\int \sin^4 x dx$

**Answers:**

(1). $\frac{(4-2x)^{-2}}{4}$ (2). $\frac{-2}{15}(3-5x)^{\frac{3}{2}} + C$

(3). $\frac{x^4}{4} + \frac{3}{2}x^2 + 3\log_e |x| - \frac{1}{2x^2} + C$ (4). $\frac{e^{5x+2}}{5} + C$ (5). $\frac{-1}{4(4x+5)} + C$

(6). $\frac{5}{9}x^{9/5} - 5x^{-1/5} - \frac{5}{2}x^{4/5} + C$

(7). $x - \sin x + C$ (8). $\frac{1}{4}\left[ \frac{3\sin 2x}{2} + \frac{\sin 6x}{6} \right] + C$

(9). $\frac{1}{4}\left| \frac{3x}{2} + \frac{\sin 4x}{8} - \sin 2x \right| + C$

## 5.2 Integration by Substitution

If the integral is of the form $\int F\{f(x)\} \cdot f'(x) dx$, where $f(x)$ is an elementary/standard function, the integral can be reduced to a simpler integrable form by putting $y = f(x)$ so that $dy = f'(x) dx.$. The integral gets reduced to the form $\int F(y) dy$, which can be done by known methods or by using standard formulas.

**Note:** The following two particular cases of $\int F\{f(x)\} f'(x) dx$ are of importance, as they will be used in integrating some rational and irrational functions.

$$(i). \int \frac{f'(x)}{f(x)} dx \rightarrow \int \frac{dy}{y} = \log y \rightarrow \log f(x)$$



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$$(ii).\int \frac{f'(x)}{\sqrt{f(x)}}dx \to \int \frac{dy}{\sqrt{y}} = 2\sqrt{y} \to 2\sqrt{f(x)}$$

**Example1:** Evaluate $\int x^3 \cos\left(x^4+2\right)dx$

**Solution:** Let $\left(x^4+2\right) = u \Rightarrow x^4 = u-2$

$$4x^3dx = du$$

$$x^3dx = \frac{du}{4}$$

Now, $\int x^3 \cos\left(x^4+2\right)dx = \int \cos u \frac{du}{4} = \frac{1}{4}\int \cos u du$

$$= \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}\sin\left(x^4+2\right) + C$$

**Example2:** Evaluate $\int \frac{x}{\sqrt{1-4x^2}}dx$

**Solution:** Let $xax = \frac{\quad}{8}$

$$-8xdx = du$$

$$xdx = \frac{-du}{8}$$

$$\int \frac{x}{\sqrt{1-4x^2}}dx = \int \frac{1}{\sqrt{u}}\left(\frac{-du}{8}\right)$$

$$= \frac{-1}{8}\left(2\sqrt{u}\right) + C$$

$$= \frac{-\sqrt{1-4x^2}}{4} + C$$

**Example 3:** Evaluate $\int \frac{1}{\left(1+e^x\right)\left(1+e^{-x}\right)}dx$

**Solution:** Let $\int \frac{1}{\left(1+e^x\right)\left(1+e^{-x}\right)}dx = \int \frac{1}{\left(1+e^x\right)\left(1+\frac{1}{e^x}\right)}dx$

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$$= \int \frac{e^x}{(1+e^x)(e^x+1)} dx$$

$$= \int \frac{e^x}{(1+e^x)^2} dx$$

Put $1+e^x = u \Rightarrow e^x = u-1$

$$e^x dx = du$$

$$\int \frac{e^x}{(1+e^x)^2} dx = \int \frac{1}{u^2} du$$

$$= \int u^{-2} du$$

$$= \left[\frac{u^{-1}}{-1}\right] + C$$

$$= \frac{-1}{u} + C$$

$$= \frac{-1}{1+e^x} + C$$

**Example 4:** Evaluate $\int \frac{1}{1+\tan x} dx$

**Solution:** Let $\int \frac{1}{1+\tan x} dx = \int \frac{1}{1+\frac{\sin x}{\cos x}} dx$

$$= \int \frac{\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int \frac{2\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int \frac{\sin x + \cos x + \cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int \left(1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

Put $u = \sin x + \cos x$

$du = (\cos x - \sin x) dx$

$$= \frac{1}{2}[x] + \frac{1}{2} \int \frac{du}{u}$$



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$$=\frac{1}{2}[x]+\frac{1}{2}\log u+C$$

$$=\frac{x}{2}+\frac{1}{2}\log(\sin x+\cos x)+C$$

**Example 5:** Evaluate $\int_1^e \frac{\log x}{x}dx$

**Solution:** Let $u=\log x$

$$du=\frac{1}{x}dx$$

When $x=1, u=0$ & When $x=e, u=1$

$$\int_1^e \frac{\log x}{x}dx=\int_0^1 u\,du=\left[\frac{u^2}{2}\right]_0^1=\frac{1}{2}-0=\frac{1}{2}$$

**Example 6: Evaluate** $\int x^3\cos\left(x^4+2\right)dx$

**Solution:** Let $I=\int x^3\cos\left(x^4+2\right)dx$

Put $y=x^4+2 \Rightarrow dy=4x^3dx$

$$=\int\frac{\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int\frac{\sin x+\cos x+\cos x-\sin x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int\left(1+\frac{\cos x-\sin x}{\sin x+\cos x}\right)dx$$

$$=\frac{1}{2}\int dx+\frac{1}{2}\int\frac{\cos x-\sin x}{\sin x+\cos x}dx$$

Put $u=\sin x+\cos x$

$du=\left(\cos x-\sin x\right)dx$

$$=\frac{1}{2}[x]+\frac{1}{2}\int\frac{du}{u}$$

$$=\frac{1}{2}[x]+\frac{1}{2}\log u+C$$

$$=\frac{x}{2}+\frac{1}{2}\log(\sin x+\cos x)+C$$

**Example 7:** Evaluate $\int_1^e \frac{\log x}{x}dx$

**Solution:** Let $u=\log x$

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$$du = \frac{1}{x}dx$$

When $x = 1, u = 0$ & When $x = e, u = 1$

$$\int_1^e \frac{\log x}{x}dx = \int_0^1 udu = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

**Example 8: Evaluate** $\int x^3 \cos\left(x^4 + 2\right)dx$

**Solution:** Let $I = \int x^3 \cos\left(x^4 + 2\right)dx$

Put $y = x^4 + 2 \Rightarrow dy = 4x^3 dx$

$$= -\frac{1}{4}\sqrt{t}$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

**Example 9: Evaluate** $\int \sqrt{1+x^2}\,x^5 dx$

**Solution:** Let $I = \int \sqrt{1+x^2}\,x^5 dx$

Put $u = 1 + x^2 \Rightarrow du = 2xdx$

$$\Rightarrow I = \int \sqrt{u}\,x^4 \cdot xdx$$

$$= \int \sqrt{u}\,(u-1)^2 \frac{1}{2}du$$

$$= \frac{1}{2}\int u^{1/2}\left(u^2 + 1 - 2u\right)du$$

$$= \frac{1}{2}\int \left(u^{5/2} + u^{1/2} - 2u^{3/2}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{5/2+1}}{5/2+1} + \frac{u^{1/2+1}}{1/2+1} - 2\frac{u^{3/2+1}}{3/2+1}\right]$$

$$= \frac{1}{2}\left[\frac{u^{7/2}}{7/2} + \frac{u^{3/2}}{3/2} - 2\frac{u^{5/2}}{5/2}\right]$$

$$= \frac{u^{7/2}}{7} + \frac{u^{3/2}}{3} - \frac{2u^{5/2}}{5}$$

$$= \frac{(1+x^2)^{7/2}}{7} + \frac{(1+x^2)^{3/2}}{3} + \frac{(1+x^2)^{5/2}}{5} + C$$

**Example 10: Evaluate** $\int (1-\tan x)^n \sec^2 xdx$

**Solution:** Let $I = \int (1-\tan x)^n \sec^2 xdx$

Put $t = 1 - \tan x \Rightarrow dt = -\sec^2 xdx$

$$\Rightarrow I = -\int t^n dt$$

$$= -\frac{t^{n+1}}{n+1} + C$$

$$= -\frac{(1-\tan x)^{n+1}}{n+1} + C$$

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**Example 11: Evaluate** $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

**Solution:** Let $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Put $\quad t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}}dx$

$$\Rightarrow I = 2\int \cos t\,dt$$
$$= 2\sin t + C$$
$$= 2\sin\sqrt{x} + C$$

**Example 12: Evaluate** $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

**Solution:** Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $\quad t = xe^x \Rightarrow dt = \left[xe^x + e^x\right]dx = e^x(1+x)dx$

$$\Rightarrow I = \int \frac{dt}{\cos^2 t}$$
$$= \int \sec^2 t\,dt$$
$$= \tan t + C$$
$$= \tan\left(xe^x\right) + C$$

**Example 13: Evaluate** $\int \frac{\sin x}{(1+\cos x)^2}dx$

**Solution:** Let $I = \int \frac{\sin x}{(1+\cos x)^2}dx$

Put $\quad t = 1+\cos x \Rightarrow dt = -\sin x\,dx$

$$\Rightarrow I = \int \frac{-dt}{t^2}$$
$$= -\int t^{-2}dt$$
$$= -\frac{t^{-2+1}}{-2+1} = -\frac{t^{-1}}{-1} = \frac{1}{t}$$
$$= \frac{1}{1+\cos x} + C$$

**Example 14: Evaluate** $\int \frac{\sin(\log x)}{x}dx$

**Solution:** Let $I = \int \frac{\sin(\log x)}{x}dx$

Put $\quad y = \log x \Rightarrow dy = \frac{1}{x}dx$

$$\Rightarrow I = \int \sin y\,dy$$
$$= -\cos y$$
$$= -\cos(\log x) + C$$

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**Example 15: Evaluate** $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

**Solution:** Let $I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Put $\quad y = \tan^{-1}x \Rightarrow dy = \frac{1}{1+x^2}\,dx$

$$\Rightarrow I = \int e^y\,dy$$
$$= e^y$$
$$= e^{\tan^{-1}x} + C$$

**Example 16: Evaluate** $\int \frac{1}{\sqrt{1-x^2}\left(\sin^{-1}x\right)^2}dx$

**Solution:** Let $I = \int \frac{1}{\sqrt{1-x^2}\left(\sin^{-1}x\right)^2}dx$

Put $\quad t = \sin^{-1}x \Rightarrow dt = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\Rightarrow I = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\sin^{-1}x} + C$$

**Example 17: Evaluate** $\int \frac{\cos x}{4+\sin^2 x}\,dx$

**Solution:** Let $I = \int \frac{\cos x}{4+\sin^2 x}\,dx$

Put $\quad y = \sin x \Rightarrow dy = \cos x\,dx$

$$\Rightarrow I = \int \frac{dy}{4+y^2}$$
$$= \int \frac{dy}{y^2+2^2}dx$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C$$

**Example 18: Evaluate** $\int e^{\sin^2 x+\cos x}(\sin 2x - \sin x)dx$

**Solution:** Let $\quad I = \int e^{\sin^2 x+\cos x}(\sin 2x - \sin x)dx$

Put $\quad t = \sin^2 x + \cos x$

$$\Rightarrow dt = (2\sin x\cos x - \sin x)dx$$
$$dt = \left(\sin 2x - \sin x\right)dx$$
$$(1) \Rightarrow I = \int e^t dt = e^t = e^{\sin^2 x+\cos x} + C$$



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**Example 19: Evaluate** $\int \frac{1}{e^x+1}dx$

**Solution:** Let $I=\int \frac{1}{e^x+1}dx=\int \frac{1}{e^x\left(1+\frac{1}{e^x}\right)}dx=\int \frac{e^{-x}}{1+\frac{1}{e^x}}dx=\int \frac{e^{-x}}{1+e^{-x}}dx$ ________(1)

Put $\quad t=1+e^{-x}\Rightarrow dt=-e^{-x}dx$

$(1)\Rightarrow I=-\int \frac{dt}{t}=-\log t=-\log\left(1+e^{-x}\right)+C$

**Example 20: Evaluate** $\int \frac{\tan x}{\sec x+\cos x}dx$

**Solution:** Let $I=\int \frac{\tan x}{\sec x+\cos x}dx=\int \frac{\sin x/\cos x}{1/\cos x+\cos x}dx=\int \frac{\sin x}{1+\cos^2 x}dx$ __(1)

Put $\quad t=\cos x\Rightarrow dt=-\sin x dx$

(1) $\quad\Rightarrow I=-\int \frac{dt}{1+t^2}=-\tan^{-1}t=-\tan^{-1}(\cos x)+C$

**Example 21: Evaluate** $\int \frac{\cos^3 x}{\sqrt{\sin x}}dx$

**Solution:** Let $I=\int \frac{\cos^3 x}{\sqrt{\sin x}}dx$

Put $\quad t=\sin x\Rightarrow dt=\cos x dx$

$$(1)\Rightarrow I=\int \frac{\cos^2 x\cos x}{\sqrt{\sin x}}dx$$

$$=\int \frac{\left(1-\sin^2 x\right)\cos x}{\sqrt{\sin x}}dx$$

$$=\int \frac{\left(1-t^2\right)dt}{\sqrt{t}}$$

$$=\int \frac{1}{\sqrt{t}}dt-\int \frac{t^2}{\sqrt{t}}dt$$

$$=\int t^{\frac{-1}{2}}dt-\int t^{\frac{3}{2}}dt$$

$$=\frac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}-\frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}$$

$$=2\sqrt{t}-\frac{2}{5}t^{\frac{5}{2}}=2\sqrt{\sin x}-\frac{2}{5}\sin^{\frac{5}{2}}x+C$$

**Example 22: Evaluate** $\int (\sin x)^{\frac{5}{2}}\cos^3 x dx$

**Solution:** Let $I=\int (\sin x)^{\frac{5}{2}}\cos^3 x dx$

Put $\quad t=\sin x\Rightarrow dt=\cos x dx$

$$(1)\Rightarrow I=\int (\sin x)^{\frac{5}{2}}\cos^2 x\cos x dx$$

$$=\int (\sin x)^{\frac{5}{2}}\left(1-\sin^2 x\right)\cos x dx$$

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$$= \int t^{\frac{5}{2}} \left(1-t^2\right) dt$$

$$= \int t^{\frac{5}{2}} dt - \int t^{\frac{9}{2}} dt$$

$$= \frac{t^{5/2+1}}{5/2+1} - \frac{t^{9/2+1}}{9/2+1}$$

$$= \frac{2}{7} t^{\frac{7}{2}} - \frac{2}{11} t^{\frac{11}{2}} = \frac{2}{7} \sin^{\frac{7}{2}} x - \frac{2}{11} \sin^{\frac{11}{2}} x + C,$$

**Example 23: Evaluate** $\int \tan x dx$

**Solution:** Let $I = \int \tan x dx = \int \frac{\sin x}{\cos x} dx$

Put $u = \cos x \Rightarrow du = -\sin x dx$

$$(1) \Rightarrow I = \int -\frac{du}{u}$$

$$= -\log(u) + C$$

$$= -\log(\cos u) + C$$

**Exercise:**

(1). $\int \cot x dx$ (2). $\int \sec x dx$ (3). $\int \operatorname{cosec} x dx$

(4). $\int \frac{1}{x \cdot \log x} dx$ (5). $\int \frac{\cos x}{a + b\sin x} dx$ (6). $\frac{1}{\left(1+x^2\right)\tan^{-1} x} dx$

(7). $\int \frac{e^{2x}}{e^{2x}-1} dx$ (8). $\int \frac{\tan x}{\log(\cos x)} dx$ (9). $\int x\left(x^2+1\right)^{7/2} dx$

(10). $\int 5x\sqrt{1-2x^2}\, dx$ (11). $\int \left(e^x+5\right)^n e^x dx$ (12). $\int \frac{\left(1+\sqrt{x}\right)^n}{\sqrt{x}} dx$

(13). $\int \sqrt{\cos x} \sin^3 x dx$ (14). $\int \frac{a}{b+ce^x} dx$ (15). $\int \frac{dx}{2+e^x+e^{-x}}$

(16). $\int x^3 \tan^4 x^4 \sec^2 x^4 dx$ (17). $\int \sqrt{1+\sin^2(x-1)} \cos(x-1) dx$

(18). $\int \frac{x^{n-1}}{a+bx^n} dx$

**Answers:**

(1). $\log(\sin x)$ (2). $\log(\sec x + \tan x)$

(3). $-\log(\operatorname{cosec} x + \cot x)$ (4). $\log(\log x)$

(5). $\frac{1}{b}\log(a + b\sin x)$ (6). $\log\left(\tan^{-1} x\right)$

(7). $\frac{1}{2}\log\left(e^{2x}-1\right)$ (8). $-\log(\log \cos x)$

(9). $\frac{1}{9} t^{9/2}$ (10). $\left(1-2x^2\right)^{3/2}$

(11). $\frac{\left(e^x+5\right)^{n+1}}{n+1}$ (12). $\frac{2\left(1+\sqrt{x}\right)^{n+1}}{n+1}$

(13). $-2\left[\frac{t^3}{3} - \frac{t^7}{7}\right]$ (14). $-\frac{a}{b}\log\left(be^{-x}+c\right)$



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| | |
|---|---|
| (15). $-\dfrac{1}{1+e^{-x}}$ | (16). $\dfrac{1}{20}\tan^4 x^4 + C$ |
| (17). $\dfrac{1}{3}\left[1+\sin^2(x-1)\right]^{3/2}$ | (18). $\dfrac{1}{nb}\log\left(a+bx^n\right)$ |

## 5.3 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

For indefinite integrals, this equation becomes or we

$$\int [f(x)g'(x) + g(x)f'(x)]dx = f(x)g(x)$$

or $\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$

we can rearrange this equation as $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

**Formula for Integration by parts:**

If u and v are functions of x then $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Integrating we get

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\Rightarrow \quad uv = \int udv + \int vdu$$

$$\Rightarrow \quad \int udv = uv - \int vdu$$

This is called the formula for integration by parts. The success of the method depends on the proper choice of u as that function which comes first in the word "ILATE" where

I - Inverse circular function
L - Logarithmic function
A - Algebraic function
T - Trigonometric function
E - Exponential function

**Example:1** Find $\int x\sin x\,dx$.

**Solution:** Here direct and substitution methods are not applicable.

$u = x\,; dv = \sin x\,dx \Rightarrow du = dx\,; v = \int \sin x\,dx \Rightarrow v = -\cos x$

Using Integration by parts

$\int u\,dv = u\,v - \int v\,du \Rightarrow \int x\sin x\,dx = x(-\cos x) - \int(-\cos x)\,dx$

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$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c$$

**Example: 2** Find $\int \log x\,dx$

**Solution:** Here direct and substitution methods are not applicable.

$$u = \log x\ ; dv = dx \Rightarrow du = \frac{1}{x}dx\ ; v = \int dx \Rightarrow v = x$$

Using Integration by parts

**Example:3** Calculate $\int_0^1 \tan^{-1} x\,dx$ .

**Solution:** Here direct and substitution methods are not applicable.

$$u = \tan^{-1} x\ ; dv = dx \Rightarrow du = \frac{1}{1+x^2}dx\ ; v = \int dx \Rightarrow v = x$$

Using Integration by parts

$$\int u\,dv = uv - \int v\,du \Rightarrow \int_0^1 \tan^{-1} x\,dx = \left(x\tan^{-1} x\right)_0^1 - \int_0^1 \frac{x}{1+x^2}dx$$

$$= 1.\tan^{-1}(1) - 0\tan^{-1}(0) - \int_0^1 \frac{x}{1+x^2}dx$$

$$= \frac{\pi}{4} - 0 - I_1 \left(\text{where } I_1 = \int_0^1 \frac{x}{1+x^2}dx\right)$$

To evaluate this $I_1 = \int_0^1 \frac{x}{1+x^2}dx$ by using substitution method

$$t = 1 + x^2\ ; dt = 2x\,dx \text{ (or) } x\,dx = \frac{dt}{2}$$

When $x = 0\ ; t = 1;$ and $x = 1\ ; t = 2;$

$$I_1 = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}[\log t]_1^2$$

$$= \frac{1}{2}[\log 2 - \log 1] = \frac{1}{2}\log 2$$

$$\boxed{\therefore \int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{1}{2}\log 2}$$

**Example:4** Evaluate $\int xe^x dx.$

**Solution:** Let $I = \int xe^x dx$

Take $u = x,\quad dv = e^x dx$

$\therefore\quad du = dx,\quad v = \int e^x dx = e^x$

$\therefore\quad I = xe^x - \int e^x \cdot dx = xe^x - e^x + c$



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**Example: 5 Evaluate** $\int e^x \sin x dx$.

**Solution:** Let $I = \int e^x \sin x dx$.

Take $u = \sin x, \quad dv = e^x dx$ (according to ILATE)

$\therefore \quad du = \cos x dx, \quad v = e^x$

$\therefore \quad I = \sin x \cdot e^x - \int e^x \cos x dx$

Now take $u = \cos x, \quad dv = e^x dx$

$du = -\sin x dx, \; v = e^x$

$$\therefore \quad I = e^x \sin x - \left[\cos x . e^x - \int e^x(-\sin x) dx\right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$= e^x \sin x - e^x \cos x - I$$

$\therefore \; 2I = e^x \sin x - e^x \cos x$

$$I = \frac{e^x}{2}(\sin x - \cos x) + c$$

**Example:6 Evaluate** $\int x \sin \; dx$.

**Solution:** Let $I = \int x \sin^2 dx$.

$$= \int x \frac{(1-\cos 2x)}{2} dx$$

$$= \frac{1}{2}\int x dx - \frac{1}{2}\int x \cos 2x dx$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{4}\left[x \sin 2x + \frac{\cos 2x}{2}\right] + c$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + c$$

**Example:7 Evaluate** $\int x \sin^2 dx$.

**Solution:** Let $I = \int x \tan^{-1} x dx$. [according to ILATE]

Take $u = \tan^{-1} x, \quad dv = dx$

$$du = \frac{1}{1+x^2} dx, \qquad v = \int x dx = \frac{x^2}{2}$$



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$$I = \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$$

**Example: 8** Evaluate $\int \frac{x^3 e^{x^2}}{(x^2+1)^2} dx$

**Solution:** Let $I = \int \frac{x^3 e^{\ \ 2}}{(x^2+1)^2} dx$

Put $\quad t = x^2 \quad \therefore dt = 2xdx \Rightarrow \frac{1}{2} dt = xdx$

$$\therefore \qquad I = \int \frac{x^2 \cdot x e^{x^2} dx}{(x^2+1)^2}$$

$$= \int \frac{te^t \frac{1}{2} dt}{(t+1)^2}$$

$$= \frac{1}{2} \int \frac{te^t}{(t+1)^2} dt$$

$$= \frac{1}{2} \int e^t \frac{[t+1-1]}{(t+1)^2} dt$$

$$= \frac{1}{2} \int e' \left( \frac{1}{t+1} - \frac{1}{(t+1)^2} \right) dt$$

If $f(t) = \frac{1}{1+t}$, then $f'(t) = -\frac{1}{(t+1)^2}$

$$I = \frac{1}{2} \int e'[f(t) + f'(t)] dt$$

$$= \frac{1}{2} e^t f(t) + c \ [\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + c]$$

$$= \frac{1}{2} e^{x^2} \cdot \frac{1}{x^2+1} + c$$

**Example: 9** Evaluate $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b sin bx]$

**Solution:** Let $I = \int e^{ax} \cos bx dx \qquad (1)$

Taking $\qquad u = e^{ax}, \qquad u' = ae^{ax}$

And $\qquad v = \cos bx, \qquad v_1 = \int v dx = \int \cos bx dx = \frac{\sin bx}{b}$



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$$I = uv_1 - \int u' v_1 dx$$

$$= e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx$$

$$= e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[e^{ax}\left(-\frac{\cos bx}{b}\right) - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx\right]$$

[Again integration by parts]

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bxdx$$

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$\Rightarrow \left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b^2}[be^{ax}\sin bx + ae^{ax}\cos bx]$$

$$\Rightarrow \left(\frac{a^2 + b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}[a\cos bx + b\sin bx]$$

$$(a^2 + b^2)I = e^{ax}[a\cos bx + b\sin bx]$$

$$I = \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx]$$

$$\therefore \int e^{ax}\cos bxdx = \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx]$$

$$i.e., \int e^{ax}\cos bxdx = \frac{e^{ax}}{a^2 + b^2}\left[a\cos bx - \frac{d}{dx}(\cos bx)\right]$$

**Example:10** Evaluate $\int e^{ax}\sin bxdx = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx]$

**Solution:** To prove $\int e^{ax}\sin bxdx = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx]$

Let $I = \int e^{ax}\sin bxdx$

And $v = \cos bx,\qquad v_1 = \int vdx = \int \cos bxdx = \frac{\sin bx}{b}$

$$I = uv_1 - \int u' v_1 dx$$

$$= e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx$$

$$= e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[e^{ax}\left(-\frac{\cos bx}{b}\right) - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx\right]$$



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[Again integration by parts]

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx dx$$

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$\Rightarrow \left(1+\frac{a^2}{b^2}\right)I = \frac{1}{b^2}[be^{ax}\sin bx + ae^{ax}\cos bx]$$

$$\Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}[a\cos bx + b\sin bx]$$

$$(a^2+b^2)I = e^{ax}[a\cos bx + b\sin bx]$$

$$I = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\therefore \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - \frac{d}{dx}(\cos bx)]$$

**Exercise:**

1. $\int_1^2 x^4 (Inx)^2\, dx$ 2. $\int \cosh 3x \cos 4x dx$

3. $\int e^{2x}(x+1)^2\, dx$

4. $\int e^{2x+3}\sin(3x+1)dx$ 5. $\int x^2 e^x dx$

6. $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right)dx$

**Answers:**

1. $\frac{32}{5}(In2)^2 - \frac{64}{25}(In2) + \frac{62}{125}$ 2. $\frac{3}{25}\cos 4x \sinh 3x + \frac{4}{25}\sin 4x \cosh 3x + c$

3. $\frac{e^{2x}}{4}(2x^2+2x+1)$ 4. $\frac{1}{13}e^{2x+3}\left[2\sin(3x+1) - 3\cos(3x+1)\right]$

5. $x^2e^x - 2xe^x + 2e^x$ 6. $e^x \tan\frac{x}{2} + c$

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### 5.4 Area Under the Curve

The area under a curve can be expressed using definite integration. If we have a function y = f(x) and we want to find the area under the curve between two x values, say x = a and x = b, we can use the definite integral:

A = ∫[a,b] f(x) dx

Here, the symbol ∫ represents the integral sign, and [a,b] indicates that we are integrating with respect to x from a to b. The result of this integration gives us the area under the curve between x = a and x = b.

**Example 1**

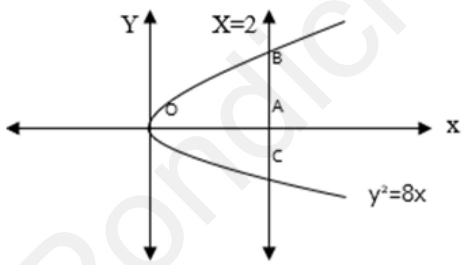
Using integration, find the area of the region bounded between the line x = 2 and the parabola $y^2 = 8x$.

Solution:

From the question it is given that two equations,

$$x = 2 - - - - - - - - -(i)$$
$$y^2 = 8x - - - - - - - -(ii)$$

So, equation (i) represents a line parallel to $y$-axis and equation (ii) represents a parabola with vertex at origin and $x$-axis as it axis, is as shown in the rough sketch below,




Now, we have to find the area of OCBO, Then, the area can be found by taking a small slice in each region of width $\Delta x$, And length = (y-0)=y

The area of sliced part will be as it is a rectangle = $y\Delta x$

So, this rectangle can move horizontal from x=0 to x=2

The required area of the region bounded between the lines = Region OCBO

$$= 2\text{ (region OABO)} = 2\int_0^2 y\,dx$$

Given, $y^2 = 8x$

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$$y = \sqrt{8x} = 2\int_0^2 \sqrt{8x}dx = 2 \cdot 2\sqrt{2}\int_0^2 \sqrt{x}dx$$

$$= 4\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = 4\sqrt{2}\left[\frac{2}{3}x\sqrt{x}\right]_0^2$$

$$= 4\sqrt{2}\left[\left(\left(\frac{2}{3}\right)2\sqrt{2}\right)-\left(\left(\frac{2}{3}\right)0\sqrt{0}\right)\right]$$

$$= 4\sqrt{2}\left(\frac{4\sqrt{2}}{3}\right) = \frac{32}{3}$$

Therefore, the required area square $= \frac{32}{3}$ units.

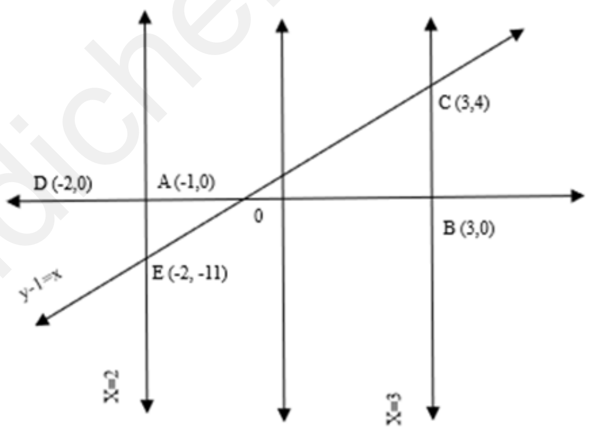
**Example 2:**

Using integration, find the area of the region bounded by the line y - 1 = x, the x-axis and the ordinates x= -2 and x = 3.

Solution:

From the question it is given that equation y - 1 = x -----------(i)

So, equation (i) represents a line that meets at $(0,1)$ and $(-1,0)$, is as shown in the rough sketch below,




Now, we have to find the area of the region bounded by the line y-1=x

So, Required area = Region ABCA + Region ADEA

$$A=\int_{-1}^{3} ydx + \int_{-2}^{-1} ydx$$

$$= \int_{-1}^{3} (x+1)dx + \int_{-2}^{-1} (x+1)dx$$

$$= \left(\frac{x^2}{2}+x\right)_{-1}^{3} + \left(\frac{x^2}{2}+x\right)_{-2}^{-1}$$



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$$A=\int_{-1}^{3} ydx+\int_{-2}^{-1} ydx$$

$$=\int_{-1}^{3}(x+1)dx+\int_{-2}^{-1}(x+1)dx$$

$$=\left(\frac{x^2}{2}+x\right)_{-1}^{3}+\left(\frac{x^2}{2}+x\right)_{-2}^{-1}$$

$$=\left[\left(\frac{3^2}{2}+3\right)-\left(\frac{(-1)^2}{2}+(-1)\right)\right]+\left[\left(\frac{(-1)^2}{2}+(-1)\right)-\left(\frac{(-2)^2}{2}+(-2)\right)\right]$$

$$=\left[\left(\frac{15}{2}+\frac{1}{2}\right)\right]+\left[-\frac{1}{2}\right]=8+\frac{1}{2}$$

$$A=\frac{17}{2}\text{square units}$$

**Example 3**

Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.

**Solution:**

From the question it is given that equation,

$y = 4x - x^2$

Adding 4 on both side,

$y + 4 = 4x - x^2 + 4$

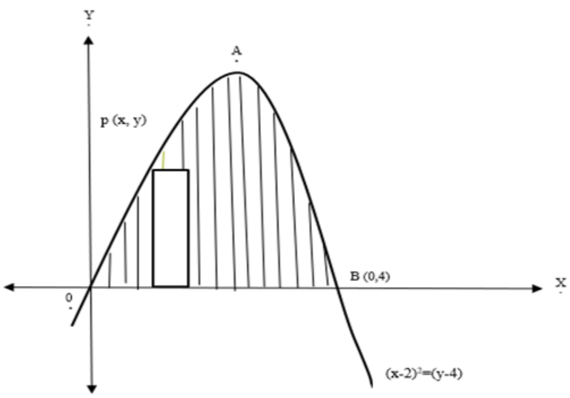
Transposing we get,

$x^2 - 4x + 4 = -y + 4$

We know that, $(a - b)^2 = a^2 - 2ab + b^2$

$(x - 2)^2 = -(y - 4)$ ... [equation (i)]

So, equation (i) represents a downward parabola with vertex $(2,4)$ and passing th $(0,0)$ and $(0,4)$, is as shown in the rough sketch below,



Then, the area can be found by taking a small slice in each region of width $\Delta x$ ,Then, the area can be found by taking a small slice in each region of width $\Delta x$, And length=(y-0)=y

The area of sliced part will be as it is a rectangle = y $\Delta x$

So, this rectangle can move horizontal from x=0 to x=a

The required area of the region bounded between the lines = Region OABO

Required area $= \text{region OABO} = \int_0^4 (4x - x^2)dx$

On integrating we get,

$$= \left(4\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^4$$

Now we have to apply limits,

= [((4x16)/2 - (64/3)] - [0-0]

On simplification we get,

= 64/6

Divide both numerator and denominator by 2 we get,

= 32/3

Therefore, the required area is $32/3$ square units.

**Example 4**

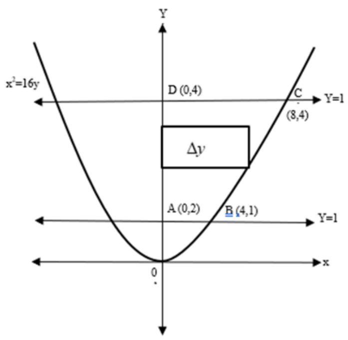
Find the area of the region bounded by x ^ 2 = 16y, y = 1, y = 4 and the y-axis in the first quadrant.

**Solution**:

From the question it is given that, Region in first quadrant bounded by y=1, y=4

Parabola $x^2 = 16y$-------------------------(i)

So, equation (i) represents a parabola with vertex $(0,0)$ and axis as $y$ - axis, as sho the rough sketch below,

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Now, we have to find the area of ABCDA,

Then, the area can be found by taking a small slice in each region of width $\Delta y$, And length =x

The area of sliced part will be as it is a rectangle = x

So, this rectangle can move horizontal from y=1 to x=4

The required area of the region bounded between the lines = Region $ABCDA$

$$= \int_1^4 x dy$$

Given, $x^2 = 16y$

Given, $x^2 = 16y$

$$x = \sqrt{(16y)}$$

$$x = 4\sqrt{x}$$

$$= \int_1^4 4\sqrt{y}dy$$

$$= 4 \cdot \left[\frac{2}{3}y\sqrt{y}\right]_1^4$$

On integrating we get,

Now, applying limits we get,

$$= 4[((2/3) \times 4 \times \sqrt{4}) - ((2/3) \times 1 \times \sqrt{1})]$$
$$= 4[(16/3) - (2/3)]$$
$$= 4[(16 - 2)/3]$$
$$= 4[14/3]$$
$$= 56/3$$

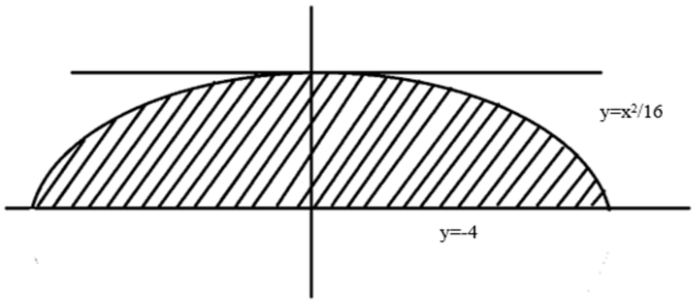
Therefore, the required area is $56/3$ square units.

**Example 5**

Find the area of the region bounded by $x^2 + 16y = 0$ and its latus rectum.

**Solution:**

We have to find the area of the region bounded by $x^2 + 16y = 0$



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Then, Area of the region

$$= 2x\int_0^8 \left[-\frac{x^2}{16} - (-4)\right] dx$$

On integrating we get,

$$= 2x\left[4x - \frac{x^3}{48}\right]_0^8$$

Now applying limits,

$$= 2 \times [(4(8-0)) - ((8)^3 - 0^3)/48)]$$
$$= 2 \times [(32) - (512/48)]$$
$$= 2 \times [(32) - (32/3)]$$
$$= 2 \times [(96 - 32)/3]$$
$$= 2 \times [64/3]$$
$$= 128/3$$

Therefore, the area of the region is $128/3$ square units.

**EXERCISE:**

Find the area of the region bounded

1. $y = x^2 + 2, y = \sin(x), x = -1$ and x=2
2. $y = \frac{8}{x}, y = 2x$ and $x = 4$
3. $x = 3 + y^2, x = 2 - y^2, y = 1$ and $y = -2$
4. $x = y^2 - y - 6$ and $x = 2y + 4$
5. $y = x\sqrt{x^2+1}, y = e^{-\frac{1}{2}x}, x = -3$ and the $y$-axis.
6. $y = 4x + 3, y = 6 - x - 2x^2, x = -4$ and $x = 2$
7. $y = \frac{1}{x+2}, y = (x+2)^2, x = -\frac{3}{2}, x = 1$
8. $x = y^2 + 1, x = 5, y = -3$ and $y = 3$
9. $x = \mathbf{e}^{1+2y}, x = \mathbf{e}^{1-y}, y = -2$ and $y = 1$

**Answers:**

| | | |
|---|---|---|
| 1. 8.0435 | 2. 6.4548 | 3. 9 |
| 4. 343/6 | 5. 17.17097 | 6. 343/12 |
| 7. 7.9695 | 8. 46/3 | 9. 22.9983 |



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**5.5 Properties of Definite Integral:**

If $f(x)$ is continuous and integrable function of $x$ in $[a,b]$, then the following properties are true.

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx; \quad a < c < b$

3. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

5. If $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ , $f(x)$ is an even function of and $\int_{-a}^{a} f(x)dx = 0$

if $f(x)$ is an odd function of $x$

6. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x) = f(x)$ and $\int_0^{2a} f(x)\, dx =$

if $f(2a-x) = -f(x)$

**Note:**

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

3) $\int_a^b cdx = c(b-a)$

4) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

**Example 1:** Evaluate $\int_1^2 \left( x^2 - 3\sqrt{x} + \frac{1}{x^2} \right) dx.$

**Solution:** $\int_1^2 \left( x^2 - 3\sqrt{x} + \frac{1}{x^2} \right) dx = \left[ \frac{x^3}{3} - 3\frac{x^{3/2}}{\frac{3}{2}} - \frac{1}{x} \right]_1^2$

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$$=\left[\left(\frac{8}{3}\right)-2\left(2\sqrt{2}\right)-\frac{1}{2}\right]-\left[\frac{1}{3}-3\right]$$

$$=\frac{7}{3}-\frac{1}{2}-\frac{1}{3}+3-4\sqrt{2}.$$

$$=\frac{29}{6}-4\sqrt{2}$$

**Example 2: Evaluate** $\int_0^{\pi/6}\cos^2\left(\frac{x}{2}\right)dx$

**Solution:** $\int_0^{\pi/6}\cos^2\left(\frac{x}{2}\right)dx=\int_0^{\pi/6}\left(\frac{1+\cos 2(x/2)}{2}\right)dx$

$$=\frac{1}{2}\int_0^{\pi/6}(1+\cos x)dx$$

$$=\frac{1}{2}[x+\sin x]_0^{\pi/6}$$

$$=\frac{\ }{2}\left[\left(\frac{\pi}{6}+\sin\left(\frac{\pi}{6}\right)\right)-\left(0+\sin(0)\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{6}+\frac{1}{2}\right]$$

**Example 3: If** $\int_0^8 f(x)dx=12,$ **and** $\int_0^8 f(x)dx=12,$ **find** $\int_8^{10} f(x)dx$

**Solution: By property,**

$$\int_0^8 f(x)dx+\int_8^{10} f(x)dx=\int_0^{10} f(x)dx$$

$$\int_8^{10} f(x)dx=\int_0^{10} f(x)dx-\int_0^8 f(x)dx=17-12=5$$

**Example 4:** $\int_0^{\pi}\log(1+\cos x)dx$

**Solution:**

$$\int_0^{\pi}\log(1+\cos x)dx$$

**let.** $I=\int_0^{\pi}\log(1+\cos x)dx$. As we know that

$$\left\{\int_0^a f(x)dx=\int_0^a f(a-x)dx\right\}$$

**Now by using the above formula we get**

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$$\Rightarrow I = \int_0^{\pi} \log(1 + \cos(\pi - x)dx$$

**Here by allied angle formula, we get**

$$\Rightarrow \mathbb{I} = \int_0^{\pi} \log(1 - \cos\ x)dx$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \{\log(1 + \cos\ x) + \log(1 - \cos\ x)\}dx$$

The above equation can be written as

$$2I = \int_0^{\pi} \log(1 - \cos^2\ x)dx$$

By using trigonometric identities, we get

$$2I = \int_0^{\pi} \log(\sin^2\ x)dx$$

$$2I = \int_0^{\pi} 2 \cdot \log(\sin\ x)dx$$

$$2I = 2 \cdot \int_0^{\pi} \log(\sin\ x)dx$$

$$I = \int_0^{\pi} \log(\sin\ x)dx \ldots\ldots (3)$$

$$\text{because. } \int_0^{2\pi} f(x)dx = 2 \cdot \int_0^{\pi} f(x)dx \cdot \text{ if } f(2a - x) = f(x)$$

Here, if $f(x) - \log(\sin\ x)$ and

$f(\pi - x) - \log(\sin(\pi - x)) = \log(\sin\ x) - f(x)$

$$\Rightarrow I = 2 \cdot \int \log\ \sin\ x dx \ldots (4)$$

$$\Rightarrow \mathrm{I} = 2 \cdot \int_0^{\frac{\pi}{2}} \log\ \sin\left(\frac{\pi}{2} - x\right)dx$$

By using trigonometric equation, we get

$$\Rightarrow I = 2. \int_0^{\frac{\pi}{2}} \log \cos x\mathrm{dx} \ldots\ldots (5)$$

Adding (1) and (2), we get



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$$\Rightarrow 2I = 2 \cdot \int_0^{\frac{\pi}{2}} (\log\ \sin\ x + \log\ \cos\ x)dx$$

Now by adding and subtracting log 2 we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\ \sin\ x + \log\ \cos\ x + \log\ 2 - \log\ 2)dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\,(2\sin\ x\cos\ x) - \log\ 2)dx$$

Now by splitting the integral we get

$$\Rightarrow \mathrm{I} = \int_0^{\frac{\pi}{2}} \Big(\log\,(\sin\ 2x)dx - \int_0^{\frac{\pi}{2}} \log\ 2dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0, t = 0$ and when $x = \pi/2, t = \pi$

$$\Rightarrow I = \left[\frac{1}{2}\int_0^{\pi} (\log\,(\sin\ t)dt\right] - \left(\frac{\pi}{2}\log\ 2\right)$$

$$\Rightarrow I = \left[\frac{1}{2}\right] - \left(\frac{\pi}{2}\log\ 2\right)$$

$$\Rightarrow I = -\left(\frac{\pi}{2}\log\ 2\right)$$

$$\Rightarrow I = -(\pi\log\ 2)$$

**Example 5 :** $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

**Solution:**

Given: $\int \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

let, $\mathrm{I} = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \ldots (1)$ As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula, we get

$$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$$



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Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

The above equation becomes,

$$\Rightarrow 2I = \int_0^a [1]dx$$

On integrating we get

$$\Rightarrow 2I = [x]_0^a$$

Now by applying the limits

$$\Rightarrow 2I = a - 0$$
$$\Rightarrow 2I = a$$
$$\Rightarrow I = \frac{a}{2}$$

**Example 6:** $\int_0^4 |x-1|dx$

**Solution:**

Given:

As we can see that $(x-1) \leq 0$ when $0 \leq x \leq 1$ and $(x-1) \geq 0$ when $1 \leq x \leq 4$

As we know that

$$\left\{\int_1^b f(x)dx = \int_1^b f(x)dx + \int_c^b f(x)dx\right\}$$

By substituting the above formula, we get

$$\Rightarrow I = \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$$

On integration

$$\Rightarrow I = -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

Now by applying the limit we get

**Exercise:** (1). $\int_0^4 (5-2t+3t^2)\,dt$ (2). $\int_1^2 (1+2\ddot{u})^2$



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(3). $\int_{0}^{\frac{\pi}{4}} \sec^2 ü$   (4). $\int_{1}^{2} \frac{v^5 - 3v^6}{v^4} dv$   ü $\int_{\frac{\pi}{2}}^{\pi} e^x \frac{ü - sinx}{ü - cosx} dx$

(6). $\int_{0}^{\frac{\pi}{4}} \frac{(sin\, x \, cos\, x)}{(cos^4 x + sin^4 x)} dx$

**Answers:**

(1). 63   (2). $\frac{49}{3}$   (3). 1   (4). $\frac{-11}{2}$   (5). $e^{\frac{\pi}{2}}$   (6). $\frac{\pi}{8}$

**Self Assesment Questions**:

(1) Evaluate $\int (\tan x - 2\cot x)^2 dx$   $[\text{Ans}: \tan x - 4\cot x - 9x + C]$

(2) Evaluate $\int e^{\tan^{-1} x} \left[ \frac{1 + x + x^2}{1 + x^2} \right] dx$   $[\text{Ans}: xe^{\tan^{-1} x} + C]$

(3) Evaluate $\int \frac{e^x ü x^3 + x +}{(x^2 + 1) 3/2} dx$   $\left[ \text{Ans}: e^x \frac{x}{(1 + x^2)^{3/2}} + c \right]$

(4) Find the area bounded by

$x = e^{ü+ \ y}, x = e^{-y}, y = -2$ and $y = 1$   $[\text{Ans}: 22.9978]$

(5) Evaluate $\int_{0}^{\frac{\pi}{4}} \frac{ü sin\, x + cos\, x}{(9 + 16 sin\, 2x)} dx$   $\left[ \text{Ans}: \frac{}{4} \log 9 \right]$

## Summary

- Integrals are mathematical tools used to find the area under a curve or the total accumulation of a function over a given interval. They are closely related to derivatives, which measure the rate of change of a function.
- There are two types of integrals: definite and indefinite. Definite integrals have specific limits of integration and give a single numerical value as a result. Indefinite integrals do not have limits and give a general formula for the antiderivative of a function.
- The process of finding integrals involves various techniques such as substitution, integration by parts, trigonometric substitution, partial

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fraction decomposition, and others. There are also special functions, such as the Gamma function and the Beta function, which play a crucial role in the evaluation of certain integrals.

1 $\int cf(x)dx = c\int f(x)dx$

2 $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

3 $\int kdx = kx + C$

4 $\int x^n dx = \frac{x^{n+1}}{n+1} + C(n \neq -1)$

5 $\int e^x dx = e^x + C$

6 $\int \frac{1}{x} dx = \ln|x| + C$

7 $\int a^x dx = \frac{a^x}{\ln a} + C$

8 $\int \sin x dx = -\cos x + C$

9 $\int \cos x dx = \sin x + C$

10 $\int \sec^2 x dx = \tan x + C$

11 $\int \csc^2 x dx = -\cot x + C$

12 $\int \sec x \tan x dx = \sec x + C$

13 $\int \csc x \cot x dx = -\csc x + C$

14 $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

15 $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

16 $\int \sinh x dx = \cosh x + C$

17 $\int \cosh x dx = \sinh x + C$

18 $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

19 $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C/\log\left(x + \sqrt{x^2-1}\right) + C$

20 $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C/\log\left(x + \sqrt{x^2+1}\right) + C$

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- Integration by Substitution:

$$\int \frac{f'(x)}{f(x)} dx \rightarrow \int \frac{dy}{y} = \log y \rightarrow \log f(x)$$

- Integration by Parts

$$\grave{O} f(x) g'(x) dx = f(x) g(x) - \grave{O} g(x) f'(x) dx$$

- Area Under the Curve

The area under a curve can be expressed using definite integration. If we have a function y = f(x) and we want to find the area under the curve between two x values, say x = a and x = b, we can use the definite integral:

A = ∫[a,b] f(x) dx

- Properties of Definite Integral:

If $f(x)$ is continuous and integrable function of $x$ in ü$a$ $b$ , then the following properties are true.

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

2. $\int_{ü}^{ü} f(x)dx = \int f(x)dx + \int f(x)dx; \quad a < c < b$

3. $\int_{-a}^{ü} f(x)dx = 2\int^{ü} f(x)dx$

4. $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

5. If $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ , $f(x)$ is an even function of $x$ and $\int_{-a}^{a} f ü x \; dx =$
if $f(x)$ is an odd function of $x$

6. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$ and $\int_0^{2a} f ü x \; dx =$ if
$f(2a ℵ x)$ $f(x)$

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**Note:**

$$1)\int_{a}^{a} f(x)dx = 0$$

$$2)\int_{\ddot{u}}^{\ddot{u}}\left[f(x) \pm g(x)\right]dx = \int \; f(x)dx \pm \int \; g(x)dx$$

$$3)\int_{a}^{b} cdx = c\left(b-a\right)$$

$$4)\int_{a}^{b} cf(x)dx = c\int_{a}^{b} f(x)dx$$

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