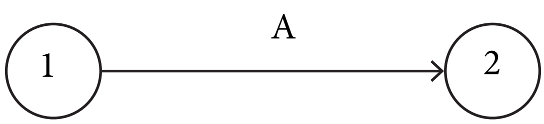
Problem 2

Develop a network diagram for the project specified below:

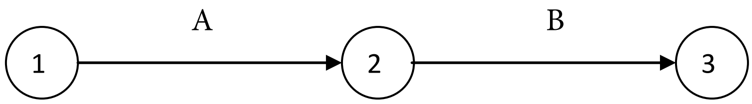
Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F

Solution

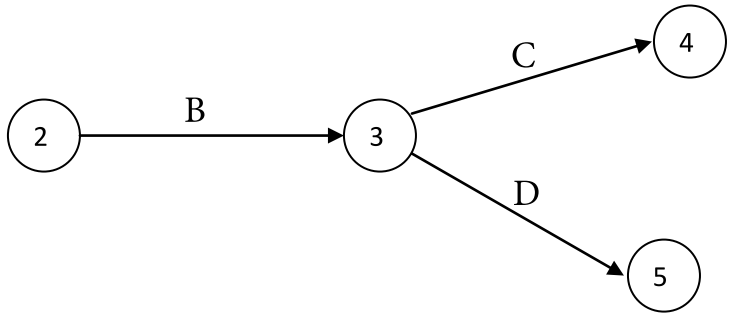
Activity A has no predecessor activity. i.e., It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2. Then we have the following representation for A:



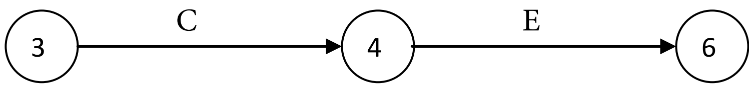
For activity B, the predecessor activity is A. Let us suppose that B joins nodes 2 and 3. Thus we get



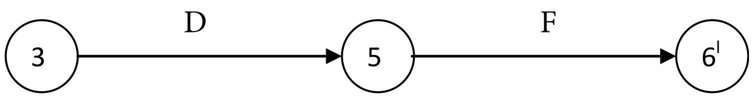
Activities C and D have B as the predecessor activity. Therefore we obtain the following:



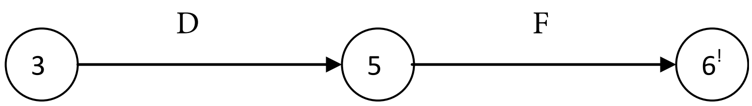
Activity E has D as the predecessor activity. So we get



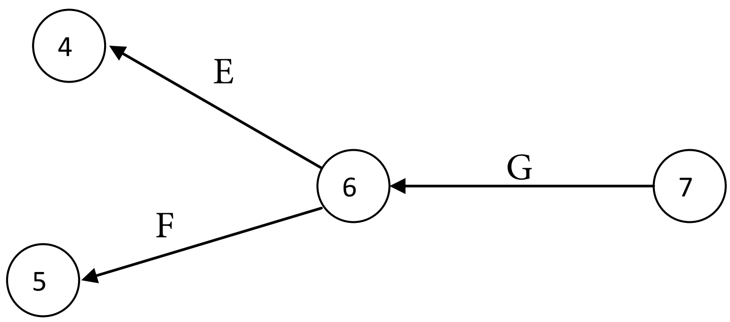
Activity F has D as the predecessor activity. So we get



Activity G has E and F as predecessor activities. This is possible only if nodes 6 and 6l are one and the same. So, rename node 6l as node 6. Then we get

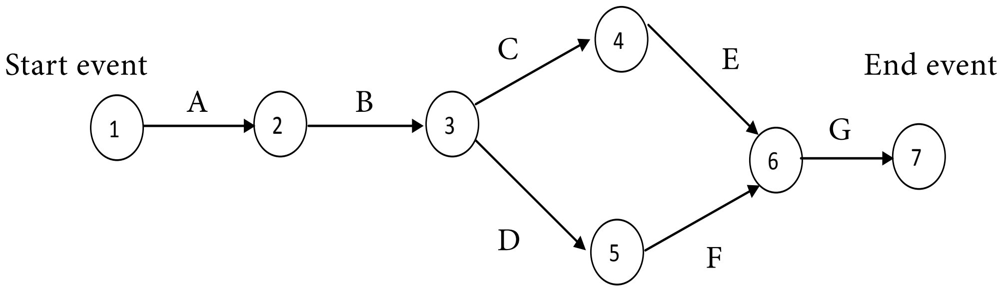


and



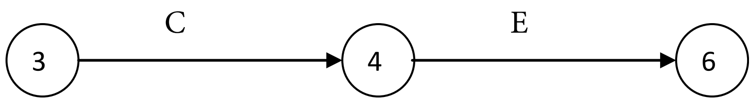
G is the last activity.

Putting all the pieces together, we obtain the following diagram the project network:

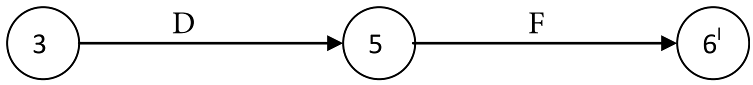


The diagram is validated by referring to the given data.

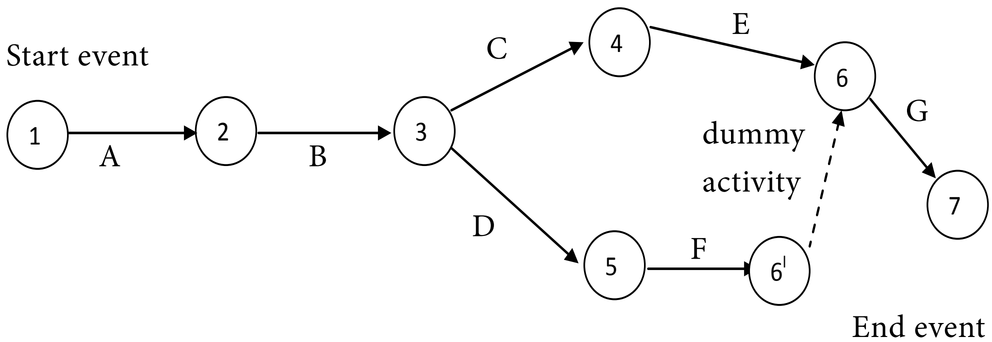
Note: An important point may be observed for the above diagram. Consider the following parts in the diagram



and



We took nodes 6 and 6l as one and the same. Instead, we can retain them as different nodes. Then, in order to provide connectivity to the network, we join nodes 6l and 6 by a dummy activity. Then we arrive at the following diagram for the project network:



Questions

1. Explain the terms: event, predecessor event, successor event, activity, dummy activity, network.
2. Construct the network diagram for the following project:

Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

****

Lesson 4 - Critical Path Method (CPM)

Lesson Outline

- The concepts of critical path and critical activities
- Location of the critical path
- Evaluation of the project completion time

Learning Objectives

After reading this lesson you should be able to

- understand the definitions of critical path and critical activities
- identify critical path and critical activities
- determine the project completion time

Introduction

The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.

Assumption For Cpm

In CPM, it is assumed that precise time estimate is available for each activity.

Project Completion Time

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

Path In A Project

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

Critical Path And Crtical Activities

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the critical path and the activities along this path are called the critical activities or bottleneck activities. The activities are called critical because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Problem 1

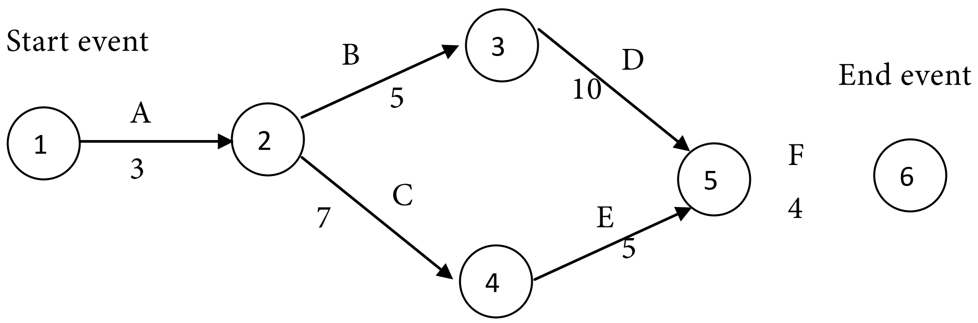
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.

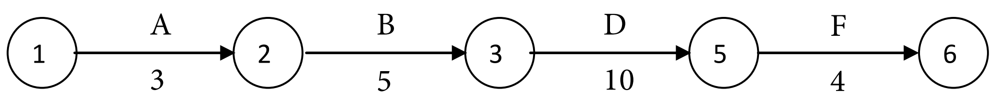
Solution

First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:



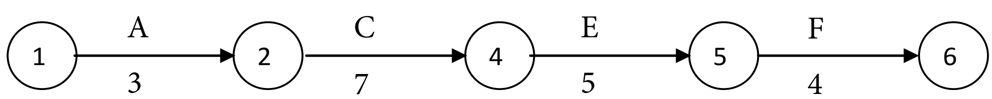
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

Path I



with a time of $3 + 5 + 10 + 4 = 22$ weeks.

Path II



with a time of $3 + 7 + 5 + 4 = 19$ weeks.

Compare the times for the two paths. Maximum of $\{22,19\} = 22$. We see that path I has the maximum time of 22 weeks. Therefore, path I is the critical path. The critical activities are A, B, D and F. The project completion time is 22 weeks.

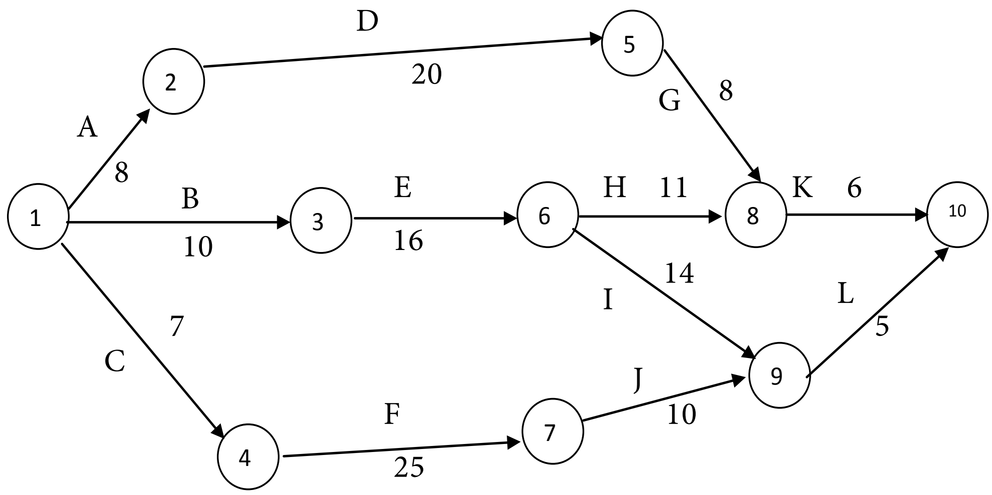
We notice that C and E are non- critical activities.

Time for path I - Time for path II = 22- 19 = 3 weeks.

Therefore, together the non- critical activities can be delayed upto a maximum of 3 weeks, without delaying the completion of the whole project.

Problem 2

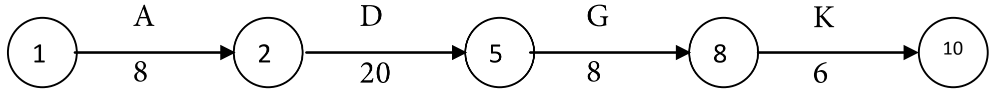
Find out the completion time and the critical activities for the following project:



Solution

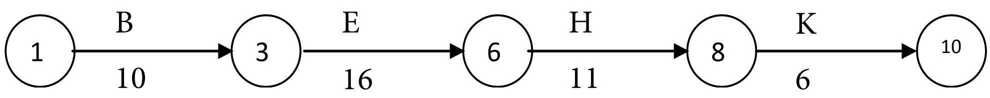
In all, we identify 4 paths, beginning with the start node of 1 and terminating at the end node of 10. They are as follows:

Path I



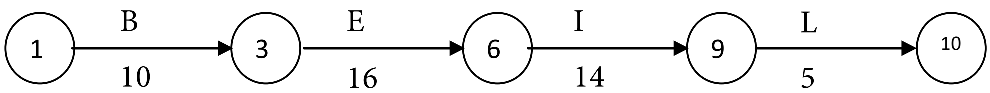
Time for the path = 8 + 20 + 8 + 6 = 42 units of time.

Path II



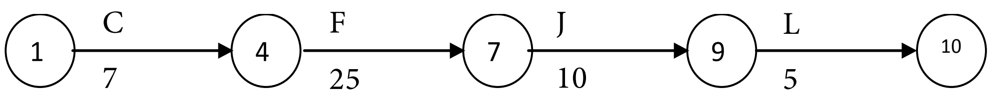
Time for the path = 10 + 16 + 11 + 6 = 43 units of time.

Path III



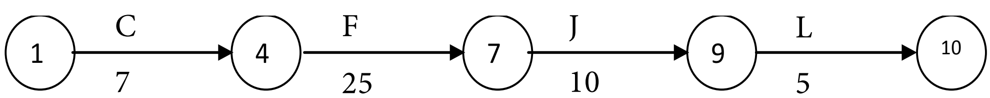
Time for the path = 10 + 16 + 14 + 5 = 45 units of time.

Path IV



Time for the path = 7 + 25 + 10 + 5 = 47 units of time.

Compare the times for the four paths. Maximum of {42, 43, 45, 47} = 47. We see that the following path has the maximum time and so it is the critical path:



The critical activities are C, F, J and L. The non-critical activities are A, B, D, E, G, H, I and K. The project completion time is 47 units of time.

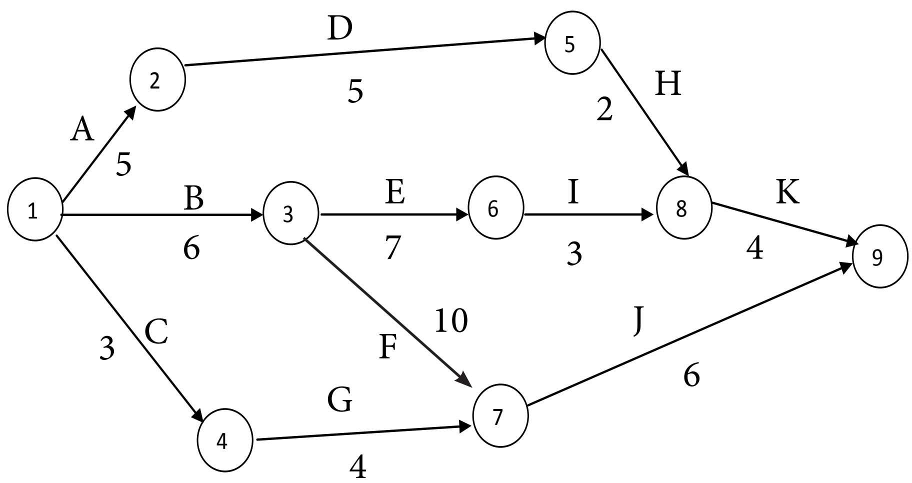
Problem 3

Draw the network diagram and determine the critical path for the following project:

Activity	Time estimate (Weeks)
1- 2	5
1- 3	6
1- 4	3
2 -5	5
3 -6	7
3 -7	10
4 -7	4
5 -8	2
6 -8	5
7 -9	6
8 -9	4

Solution

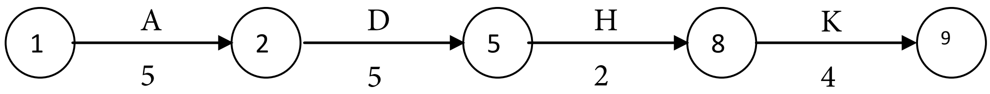
We have the following network diagram for the project:



Solution

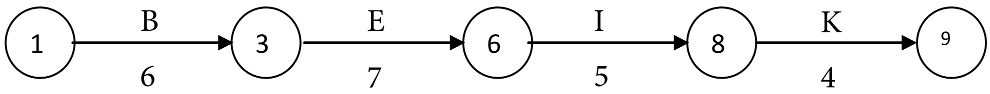
We assert that there are 4 paths, beginning with the start node of 1 and terminating at the end node of 9. They are as follows:

Path I



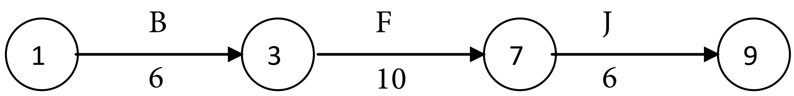
Time for the path = 5 + 5 + 2 + 4 = 16 weeks.

Path II



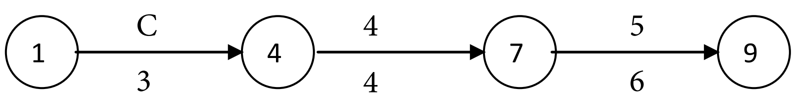
Time for the path = 6 + 7 + 5 + 4 = 22 weeks.

Path III



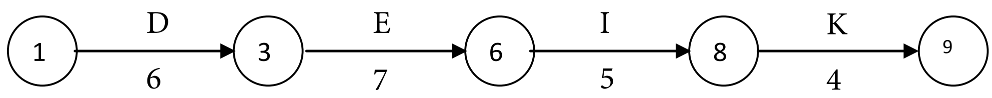
Time for the path = 6 + 10 + 6 = 16 weeks.

Path IV



Time for the path = 3 + 4 + 6 = 13 weeks.

Compare the times for the four paths. Maximum of {16, 22, 16, 13} = 22. We see that the following path has the maximum time and so it is the critical path:



The critical activities are B, E, I and K. The non-critical activities are A, C, D, F, G, H and J. The project completion time is 22 weeks.

Questions

1. Explain the terms: critical path, critical activities.
2. The following are the time estimates and the precedence relationships of the activities in a project network:

Activity	IMMEDIATE Predecessor Activity	time estimate (weeks)
A	-	4
B	-	7
C	-	3
D	A	6
E	B	4
F	B	7
G	C	6
H	E	10
I	D	3
J	F, G	4
K	H, I	2

Draw the project network diagram. Determine the critical path and the project completion time.

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Lesson 5 - Pert

Lesson Outline

- The Concept Of Pert
- Estimates Of The Time Of An Activity
- Determination Of Critical Path
- Probability Estimates
- Normal Probability Distribution Table

Learning Objectives

After reading this lesson you should be able to

- Understand the importance of PERT
- Locate the critical path
- Determine the project completion time
- Find out the probability of completion of a project before a stipulated time
- Use the normal probability distribution table

Introduction

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

Assumptions for Pert

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

Pessimistic time estimate (t_p)
Optimistic time estimate (t_o)
Most likely time estimate (t_m)

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$$t_o \le t_m \le t_p$$

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate (t_e) as the weighted average of these estimates as follows:

$$t_e = \frac{t_o + 4\ t_m + t_p}{6}$$

Since we have taken 6 units (1 for t_p , 4 for t_m and 1 for t_o), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

Measure Of Certainty

The 3 estimates of time are such that

$$t_o \le t_m \le t_p$$

Therefore the range for the time estimate is $t_p - t_o$.

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation = $\sigma = \frac{t_p - t_o}{6}$

and the variance = $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$

The certainty of the time estimate of an activity can be analysed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

Problem 1

Two experts A and B examined an activity and arrived at the following time estimates.

Expert	Time Estimate		
	t_o	t_m	t_p
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

Solution

Variance (σ^2) in time estimates = $\left(\frac{t_p - t_o}{6}\right)^2$

In the case of expert A, the variance = $\left(\frac{8-4}{6}\right)^2 = \frac{4}{9}$

As regards expert B, the variance = $\left(\frac{10-4}{6}\right)^2 = 1$

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

Determination of Project Completion Time in PERT

Problem 2

Find out the time required to complete the following project and the critical activities:

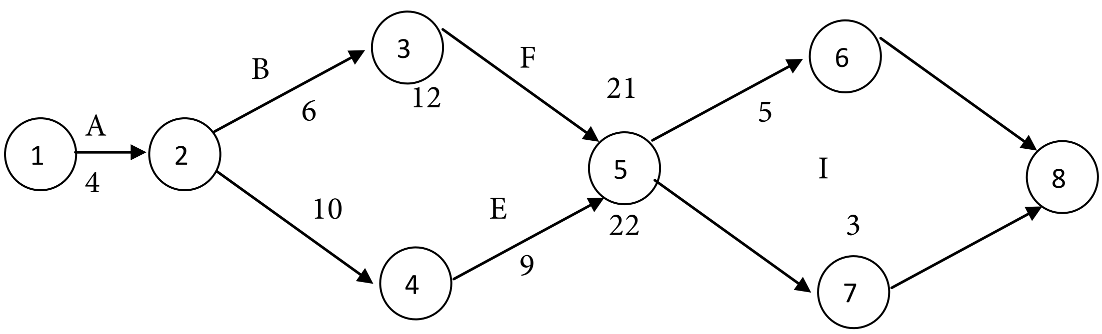
Activity	Predecessor Activity	Optimistic time estimate (to days)	Most likely time estimate (tm days)	Pessimistic time estimate (tp days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10
F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

Solution

From the three time estimates t_p, t_m and t_o, calculate t_e for each activity. We obtain the following table:

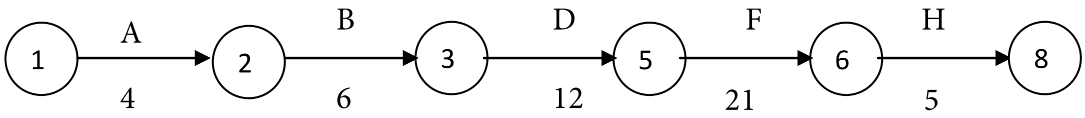
Activity	Optimistic time estimate (to)	4 x Most likely time estimate	Pessimistic time estimate (tp)	to+ 4tm + tp	Time estimate $t_e = \frac{t_o + 4\ t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21
G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Using the single time estimates of the activities, we get the following network diagram for the project.



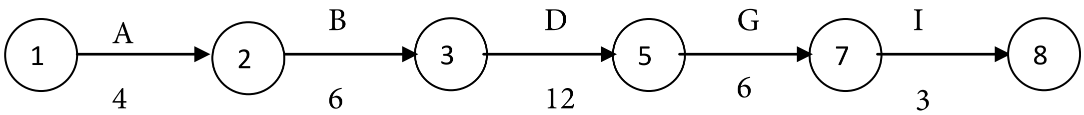
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



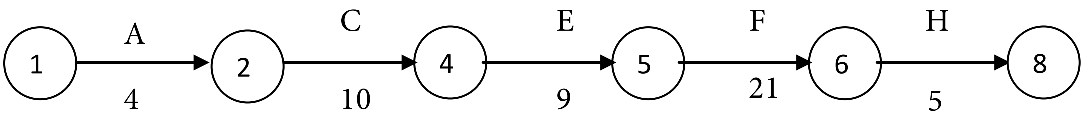
Time for the path: 4+6+12+21+5 = 48 days.

Path II



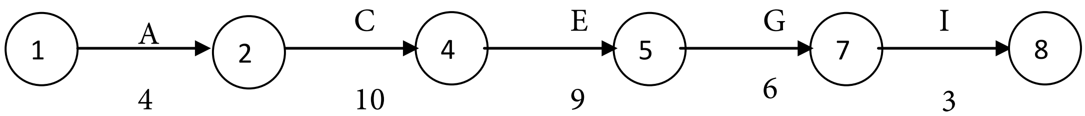
Time for the path: 4+6+12+ 6+3 = 31 days.

Path III



Time for the path: 4+10+9+ 21+5 = 49 days.

Path IV



Time for the path: 4+10+9+ 6+3 = 32 days.

Compare the times for the four paths.

Maximum of {48, 31, 49, 32} = 49.

We see that Path III has the maximum time.

Therefore the critical path is Path III. i.e., 1→2→4→5→6→8.

The critical activities are A, C, E, F and H.

The non-critical activities are B, D, G and I.

Project time (Also called project length) = 49 days.

Problem 3

Find out the time, variance and standard deviation of the project with the following time estimates in weeks:

Activity	Optimistic time estimate (to)	Most likely time estimate (tm)	Pessimistic time estimate (tp)
1-2	3	6	9
1-6	2	5	8
2-3	6	12	18
2-4	4	5	6
3-5	8	11	14
4-5	3	7	11
6-7	3	9	15
5-8	2	4	6
7-8	8	16	18

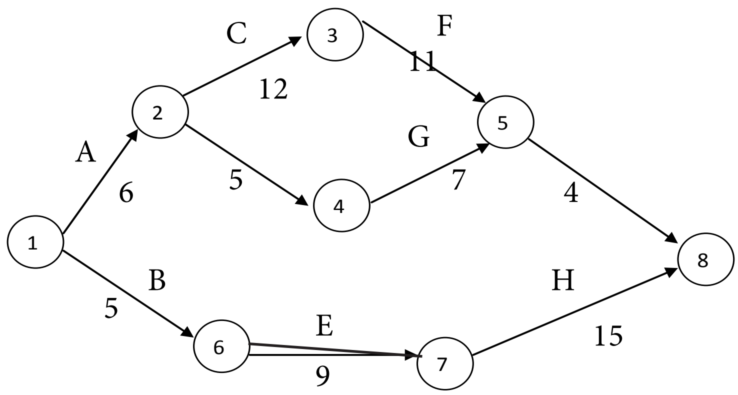
Solution

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

Activity	Optimistic time stimate (to)	4 x Most likely time estimate	Pessimistic time estimate (tp)	to+ 4tm + tp	Time estimate $t_e = \frac{t_o + 4\ t_m + t_p}{6}$
1-2	3	24	9	36	6
1-6	2	20	8	30	5
2-3	6	48	18	72	12
2-4	4	20	6	30	5
3-5	8	44	14	66	11

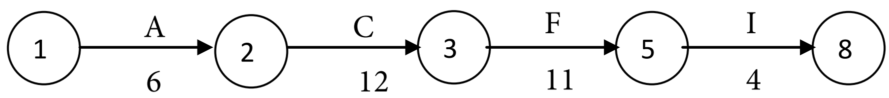
4-5	3	28	11	42	7
6-7	3	36	15	54	9
5-8	2	16	6	24	4
7-8	8	64	18	90	15

With the single time estimates of the activities, we get the following network diagram for the project.



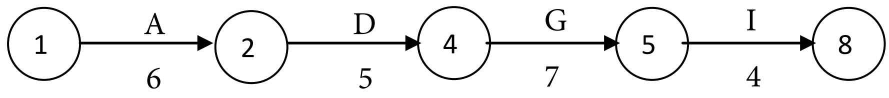
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



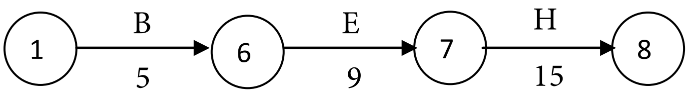
Time for the path: 6+12+11+4 = 33 weeks.

Path II



Time for the path: 6+5+7+ 4= 22 weeks.

Path III



Time for the path: 5+9+15 = 29 weeks.

Compare the times for the three paths.

Maximum of {33, 22, 29} = 33.

It is noticed that Path I has the maximum time.

Therefore the critical path is Path I. i.e., 1→2→3→5→8

The critical activities are A, C, F and I.

The non-critical activities are B, D, G and H.

Project time = 33 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (to)	Most likely timeestimate (tm)	Pessimistic time estimate (tp)	Range (tp - to)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: 1 2	3	6	9	6	1	1
C: 2 3	6	12	18	12	2	4
F: 3 5	8	11	14	6	1	1
I: 5 8	2	4	6	4	2/3	4/9

Variance of project time (Also called Variance of project length) =

Sum of the variances for the critical activities = 1+4+1+ 4/9 = 58/9 Weeks.

Standard deviation of project time = √Variance = √58/9 = 2.54 weeks.

Problem 4

A project consists of seven activities with the following time estimates. Find the probability that the project will be completed in 30 weeks or less.

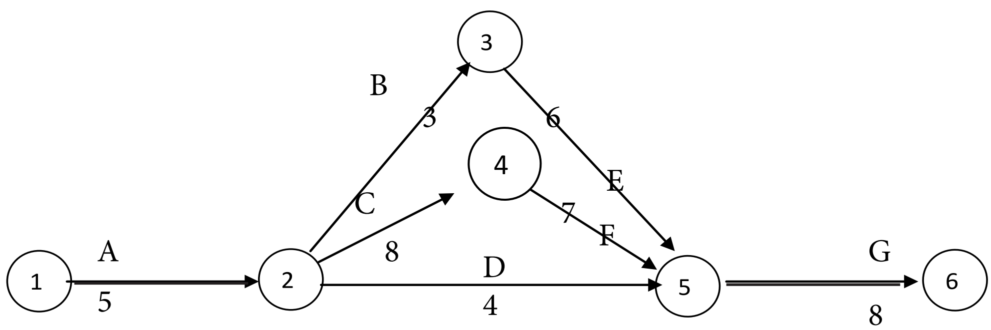
Activity	Predecessor Activity	Optimistic time estimate (to days)	Most likely time estimate (tm days)	Pessimistic time estimate (tp days)
A	-	2	5	8
B	A	2	3	4
C	A	6	8	10
D	A	2	4	6
E	B	2	6	10
F	C	6	7	8
G	D, E, F	6	8	10

Solution

From the three time estimates , and , calculate for each activity. The results are furnished in the following table:

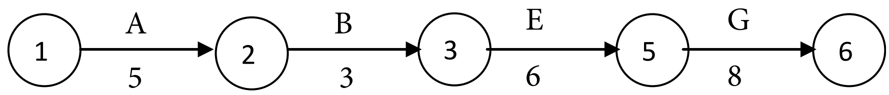
Activity	Optimistic time estimate (to)	4 x Most likely time estimate	Pessimistic time estimate (tp)	to+ 4tm + tp	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
A	2	20	8	30	5
B	2	12	4	18	3
C	6	32	10	48	8
D	2	16	6	24	4
E	2	24	10	36	6
F	6	28	8	42	7
G	6	32	10	48	8

With the single time estimates of the activities, the following network diagram is constructed for the project.



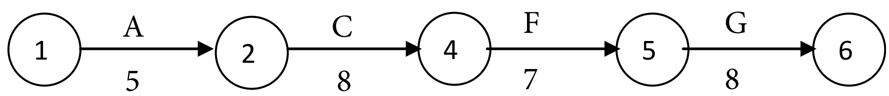
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



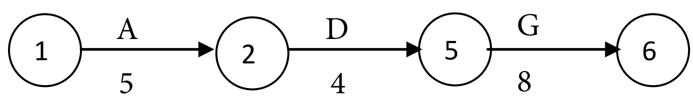
Time for the path: 5+3+6+8 = 22 weeks.

Path II



Time for the path: 5+8+7+ 8 = 28 weeks.

Path III



Time for the path: 5+4+8 = 17 weeks.

Compare the times for the three paths.

Maximum of {22, 28, 17} = 28.

It is noticed that Path II has the maximum time.

Therefore the critical path is Path II. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$.

The critical activities are A, C, F and G.

The non-critical activities are B, D and E.

Project time = 28 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (to)	Most likely time estimate (tm)	Pessimistic time estimate (tp)	Range (tp -to)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: 1 2	2	5	8	6	1	1
C: 2 4	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$
F: 4 5	6	7	8	2	$\frac{1}{3}$	$\frac{1}{9}$
G: 5 6	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$

Standard deviation of the critical path = √2 = 1.414

The standard normal variate is given by the formula

$$Z = \frac{Given\ value\ of\ t\ -\ Expected\ value\ of\ t\ \text{in}\ the\ critical\ path}{SD\ for\ the\ critical\ path}$$

So we get $Z = \frac{30 - 28}{1.414} = 1.414$

We refer to the Normal Probability Distribution Table.

Corresponding to Z = 1.414, we obtain the value of 0.4207

We get 0.5 + 0.4207 = 0. 9207

Therefore the required probability is 0.92

i.e., There is 92% chance that the project will be completed before 30 weeks.

In other words, the chance that it will be delayed beyond 30 weeks is 8%

Questions

1. Explain how time of an activity is estimated in PERT.
2. Explain the measure of certainty in PERT.
3. The estimates of time in weeks of the activities of a project are as follows:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	A	8	11	20
C	A	10	15	20
D	B	12	18	24
E	C	8	13	24
F	C	4	7	16
G	D,F	14	18	28
H	E	10	12	14
I	G,H	7	10	19

Determine the critical activities and the project completion time.

4. Draw the network diagram for the following project. Determine the time, variance and standard deviation of the project.:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	12	14	22
B	-	16	17	24
C	A	14	15	16
D	A	13	18	23
E	B	16	18	20
F	D,E	13	14	21
G	C,F	6	8	10

5. Consider the following project with the estimates of time in weeks:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	-	3	5	7
C	A	5	6	13
D	A	4	8	12
E	B,C	5	6	13
F	D,E	6	8	14

Find the probability that the project will be completed in 27 weeks.

NORMAL DISTRIBUTION TABLE

Area Under Standard Normal Distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817

2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

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Lesson 6 - Earliest And Latest Times

Lesson Outline

- The Concepts Of Earliest And Latest Times
- The Concept Of Slack
- Numerical Problems

Learning Objectives

- After reading this lesson you should be able to
- Understand the concepts of earliest and latest times
- Understand the concept of slack
- Calculate the earliest and latest times
- Find out the slacks
- Identify the critical activities
- Carry out numerical problems

Introduction

A project manager has the responsibility to see that a project is completed by the stipulated date, without delay. Attention is focused on this aspect in what follows.

Key concepts

Certain key concepts are introduced below.

Earliest Times Of An Activity

We can consider (i) Earliest Start Time of an activity and (ii) Earliest Finish Time of an activity.

Earliest Start Time of an activity is the earliest possible time of starting that activity on the condition that all the other activities preceding to it were began at the earliest possible times.

Earliest Finish Time of an activity is the earliest possible time of completing that activity. It is given by the formula.

The Earliest Finish Time of an activity = The Earliest Start Time of the activity + The estimated duration to carry out that activity.

LATEST TIMES OF AN ACTIVITY

We can consider (i) Latest Finish Time of an activity and (ii) Latest Start Time of an activity.

Latest Finish Time of an activity is the latest possible time of completing that activity on the condition that all the other activities succeeding it are carried out as per the plan of the management and without delaying the project beyond the stipulated time.

Latest Start Time of an activity is the latest possible time of beginning that activity. It is given by the formula

Latest Start Time of an activity = The Latest Finish Time of the activity - The estimated duration to carry out that activity.

TOTAL FLOAT OF AN ACTIVITY

Float seeks to measure how much delay is acceptable. It sets up a control limit for delay.

The total float of an activity is the time by which that activity can be delayed without delaying the whole project. It is given by the formula

Total Float of an Activity = Latest Finish Time of the activity - Earliest Finish Time of that activity.

It is also given by the formula

Total Float of an Activity = Latest Start Time of the activity - Earliest

Start Time of that activity.

Since a delay in a critical activity will delay the execution of the whole project, the total float of a critical activity must be zero.

EXPECTED TIMES OF AN EVENT

An event occurs at a point of time. We can consider (i) Earliest Expected Time of Occurrence of an event and (ii) Latest Allowable Time of Occurrence an event.

The Earliest Expected Time of Occurrence of an event is the earliest possible time of expecting that event to happen on the condition that all the preceding activities have been completed.

The Latest Allowable Time of Occurrence of an event is the latest possible time of expecting that event to happen without delaying the project beyond the stipulated time.

Procudure To Find The Earliest Expected Time Of An Event

Step 1

Take the Earliest Expected Time of Occurrence of the Start Event as zero.

Step 2

For an event other than the Start Event, find out all paths in the network which connect the Start node with the node representing the event under consideration.

Step 3

In the "Forward Pass" (i.e., movement in the network from left to right), find out the sum of the time durations of the activities in each path identified in Step 2.

Step 4.

The path with the longest time in Step 3 gives the Earliest Expected Time of Occurrence of the event

Working Rule for finding the earliest expected time of an event:

For an event under consideration, locate all the predecessor events and identify their earliest expected times. With the earliest expected time of each event, add the time duration of the activity connecting that event to the event under consideration. The maximum among all these values gives the Earliest Expected Time of Occurrence of the event.

Procudure To Find The Latest Allowable Time Of An Event

We consider the "Backward Pass" (i.e., movement in the network from right to left).

The latest allowable time of occurrence of the End Node must be the time of completion of the project. Therefore it shall be equal to the time of the critical path of the project.

Step 1

Identify the latest allowable time of occurrence of the End Node.

Step 2

For an event other than the End Event, find out all paths in the network which connect the End node with the node representing the event under consideration.

Step 3

In the "Backward Pass" (i.e., movement in the network from right to left), subtract the time durations of the activities along each such path.

Step 4

The Latest Allowable Time of Occurrence of the event is determined by the path with the longest time in Step 3. In other words, the smallest value of time obtained in Step 3 gives the Latest Allowable Time of Occurrence of the event.

Working Rule for finding the latest allowable time of an event:

For an event under consideration, locate all the successor events and identify their latest allowable times. From the latest allowable time of each successor event, subtract the time duration of the activity that begins with the event under consideration. The minimum among all these values gives the Latest Allowable Time of Occurrence of the event.

Slack Of An Event

The allowable time gap for the occurrence of an event is known as the slack of that event. It is given by the formula

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

Slack Of An Activity

The slack of an activity is the float of the activity.

Problem 1

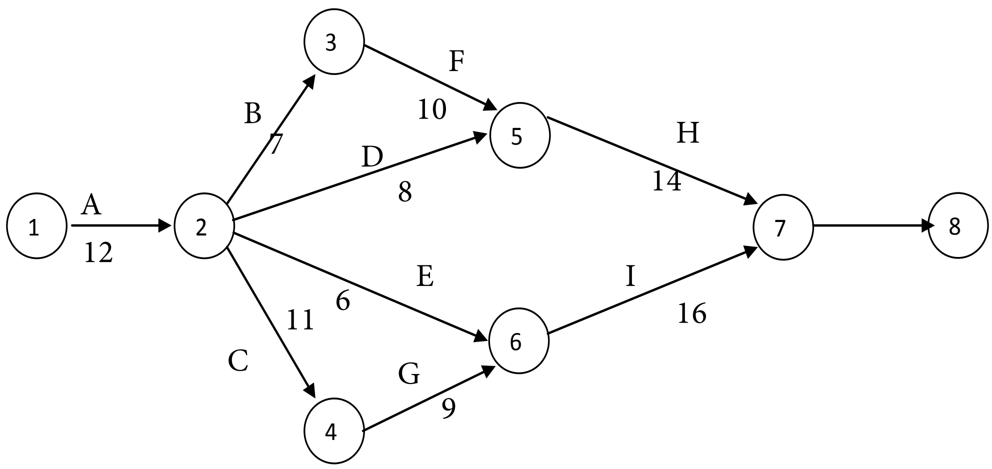
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	12
B	A	7
C	A	11
D	A	8
E	A	6
F	B	10
G	C	9
H	D, F	14
I	E, G	13
J	H, I	16

Determine the earliest and latest times, the total float for each activity, the critical activities and the project completion time.

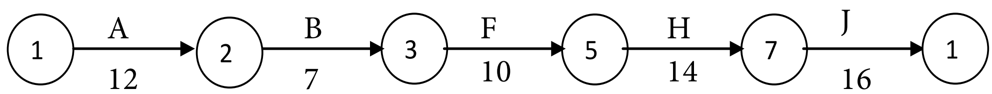
Solution

With the given data, we construct the following network diagram for the project.



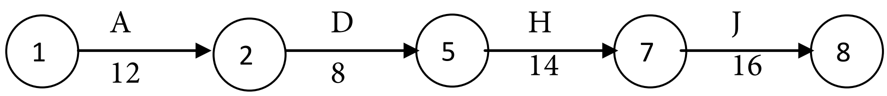
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



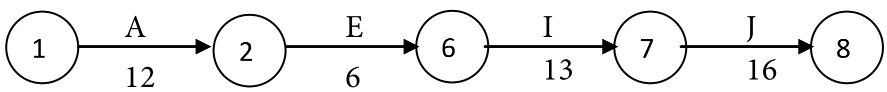
Time of the path = 12 + 7 + 10 + 14 + 16 = 59 weeks.

Path II



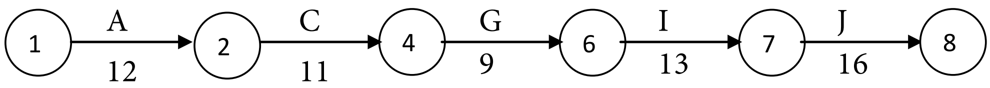
Time of the path = 12 + 8 + 14 + 16 = 50 weeks.

Path III



Time of the path = 12 + 6 + 13 + 16 = 47 weeks.

Path IV



Time of the path = 12 + 11 + 9 + 13 + 16 = 61 weeks.

Compare the times for the four paths. Maximum of {51, 50, 47, 61} = 61. We see that the maximum time of a path is 61 weeks.

Forward pass

Calculation of Earliest Expected Time of Occurrence of Events

Node	**Earliest Time of Occurrence of Node**
1	0
2	Time for Node 1 + Time for Activity A = 0 + 12 = 12
3	Time for Node 2 + Time for Activity B = 12 + 7 = 19
4	Time for Node 2 + Time for Activity C = 12 + 11 = 23
5	Max {Time for Node 2 + Time for Activity D, Time for Node 3 + Time for Activity F} = Max {12 + 8, 19 + 10} = Max {20, 29} = 29
6	Max {Time for Node 2 + Time for Activity E, Time for Node 4 + Time for Activity G} = Max {12 + 6, 23 + 9} = Max {18, 32} = 32
7	Max {Time for Node 5 + Time for Activity H, Time for Node 6 + Time for Activity I} = Max {29 + 14, 32 + 13} = Max {43, 45} = 45
8	Time for Node 7 + Time for Activity J = 45 + 16 = 61

Using the above values, we obtain the Earliest Start Times of the activities as follows:

Activity	Earliest Start Time (Weeks)
A	0
B	12
C	12
D	12
E	12
F	19
G	23
H	29
I	32
J	45

Backward pass

Calculation of Latest Allowable Time of Occurrence of Events

Node	**Latest Allowable Time of Occurrence of Node**
8	Maximum time of a path in the network = 61
7	Time for Node 8 - Time for Activity J = 61 -16 = 45
6	Time for Node 7 - Time for Activity I = 45 -13 = 32
5	Time for Node 7 - Time for Activity H = 45 -14 = 31
4	Time for Node 6 - Time for Activity G = 32 - 9 = 23
3	Time for Node 5 - Time for Activity F = 31- 10 = 21

2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C, Time for Node 5 - Time for Activity D, Time for Node 6 - Time for Activity E} = Min {21 - 7, 23 - 11, 31 - 8, 32 - 6} = Min {14, 12, 23, 26} = 12
1	Time for Node 2 - Time for Activity A = 12- 12 = 0

Using the above values, we obtain the Latest Finish Times of the activities as follows:

Activity	Latest Finish Time (Weeks)
J	61
I	45
H	45
G	32
F	31
E	32
D	31
C	23
B	21
A	12

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	12	0	12	0	12	0
B	7	12	19	14	21	2
C	11	12	23	12	23	0
D	8	12	20	23	31	11
E	6	12	18	26	32	14
F	10	19	29	21	31	2
G	9	23	32	23	32	0
H	14	29	43	31	45	2
I	13	32	45	32	45	0
J	16	45	61	45	61	0

The activities with total float = 0 are A, C, G, I and J. They are the critical activities.

Project completion time = 61 weeks.

Problem 2

The following are the details of the activities in a project:

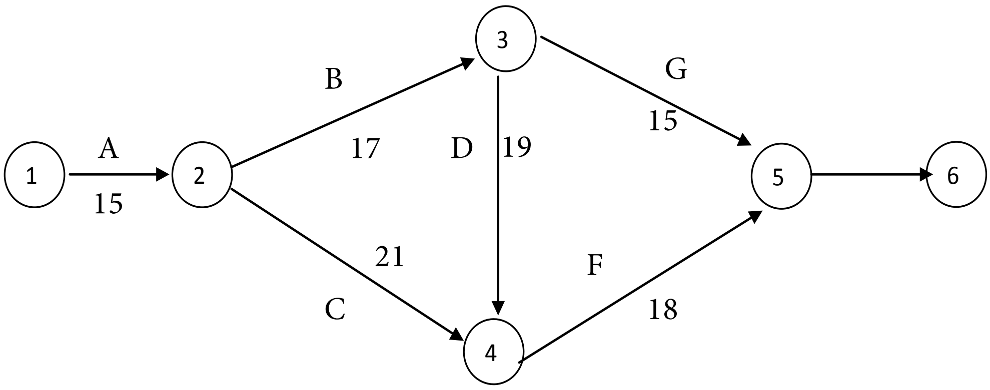
Activity	Predecessor Activity	Duration (Weeks)
A	-	15
B	A	17
C	A	21

D	B	19
E	B	22
F	C, D	18
G	E, F	15

Calculate the earliest and latest times, the total float for each activity and the project completion time.

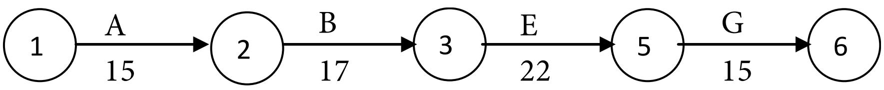
Solution

The following network diagram is obtained for the given project.



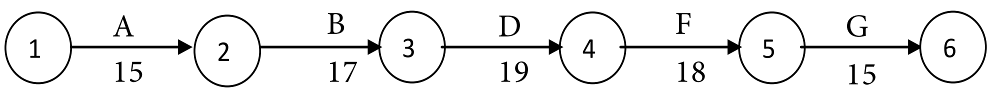
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



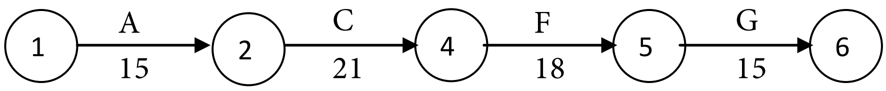
Time of the path = 15 + 17 + 22 + 15 = 69 weeks.

Path II



Time of the path = 15 + 17 + 19 + 18 + 15 = 84 weeks.

Path III



Time of the path = 15 + 21 + 18 + 15 = 69 weeks.

Compare the times for the three paths. Maximum of {69, 84, 69} = 84. We see that the maximum time of a path is 84 weeks.

Forward pass

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 15 = 15
3	Time for Node 2 + Time for Activity B = 15 + 17 = 32
4	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity D} = Max {15 + 21, 32 + 19} = Max {36, 51} = 51
5	Max {Time for Node 3 + Time for Activity E, Time for Node 4 + Time for Activity F} = Max {32 + 22, 51 + 18} = Max {54, 69} = 69
6	Time for Node 5 + Time for Activity G = 69 + 15 = 84

Calculation of Earliest Time for Activities

Activity	Earliest Start Time (Weeks)
A	0
B	15
C	15
D	32
E	32
F	51
G	69

Backward pass

Calculation of the Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
6	Maximum time of a path in the network = 84
5	Time for Node 6 - Time for Activity G = 84 -15 = 69
4	Time for Node 5 - Time for Activity F = 69 - 18 = 51
3	Min {Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity E} = Min {51 - 19, 69 - 22} = Min {32, 47} = 32
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C} = Min {32 - 17, 51 - 21} = Min {15, 30} = 15
1	Time for Node 2 - Time for Activity A = 15 - 15 = 0

Calculation of the Latest Finish Times of the activities

Activity	**Latest Finish Time (Weeks)**
G	84
F	69
E	69
D	51
C	51
B	32
A	15

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	15	0	15	0	15	0
B	17	15	32	15	32	0
C	21	15	36	30	51	15
D	19	32	51	32	51	0
E	22	32	54	47	69	15
F	18	51	69	51	69	0
G	15	69	84	69	84	0

The activities with total float = 0 are A, B, D, F and G. They are the critical activities.
Project completion time = 84 weeks.

Problem 3

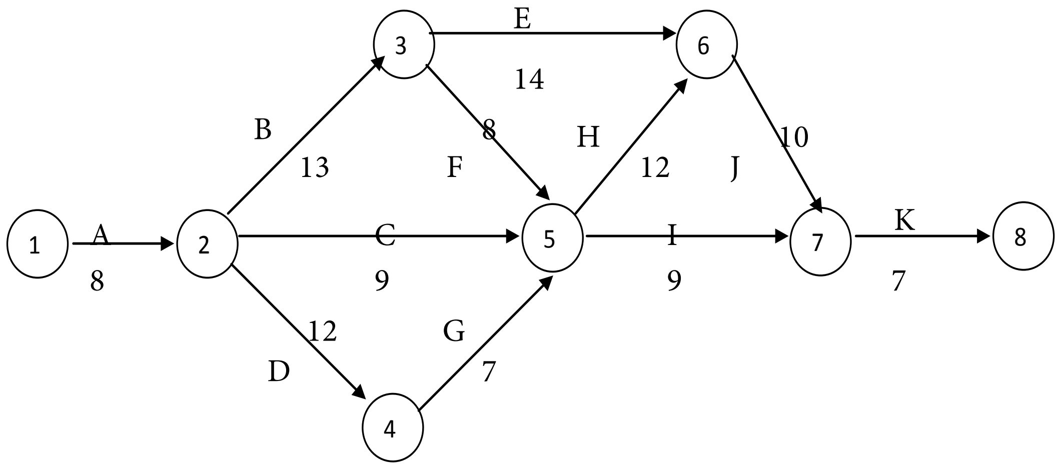
Consider a project with the following details:

Name of Activity	Predecessor Activity	Duration (Weeks)
A	-	8
B	A	13
C	A	9
D	A	12
E	B	14
F	B	8
G	D	7
H	C, F, G	12
I	C, F, G	9
J	E, H	10
K	I, J	7

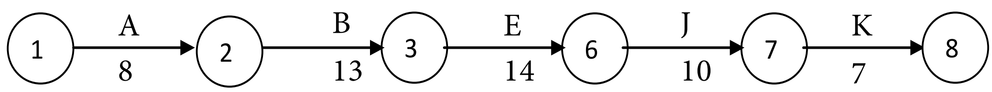
Determine the earliest and latest times, the total float for each activity, the critical activities, the slacks of the events and the project completion time.

Solution

The following network diagram is got for the given project:

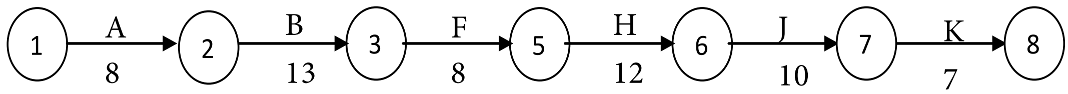


Path I



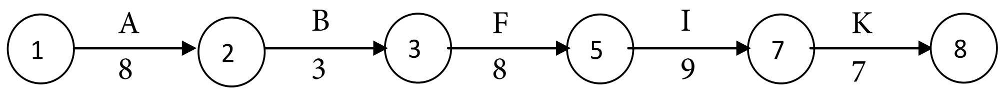
Time of the path = 8 + 13 + 14 + 10 + 7 = 52 weeks.

Path II



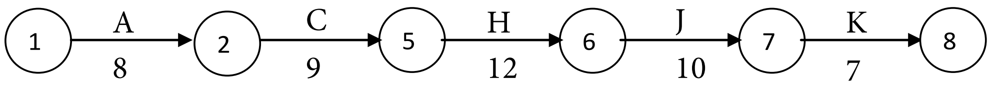
Time of the path = 8 + 13 + 8 + 12 + 10 + 7 = 58 weeks.

Path III



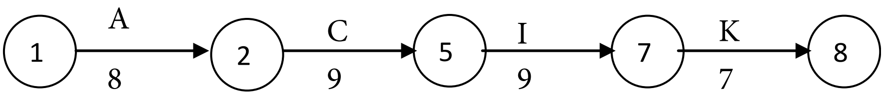
Time of the path = 8 + 13 + 8 + 9 + 7 = 45 weeks.

Path IV



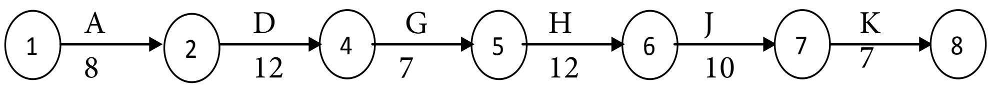
Time of the path = 8 + 9 + 12 + 10 + 7 = 46 weeks.

Path V



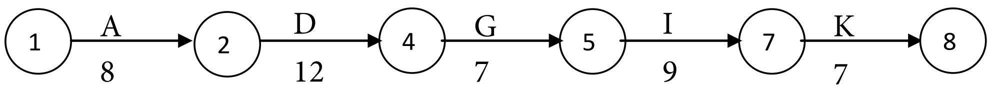
Time of the path = 8 + 9 + 9 + 7 = 33 weeks.

Path VI



Time of the path = 8 + 12 + 7 + 12 + 10 + 7 = 56 weeks.

Path VII



Time of the path = 8 + 12 + 7 + 9 + 7 = 43 weeks.

Compare the times for the three paths. Maximum of {52, 58, 45, 46, 33, 56, 43} = 58.

We see that the maximum time of a path is 58 weeks.

Forward pass

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 8 = 8
3	Time for Node 2 + Time for Activity B = 8 + 13 = 21
4	Time for Node 2 + Time for Activity D = 8+ 12 = 20
5	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity F, Time for Node 4 + Time for Activity G} = Max { 8 + 9, 21 + 8 , 20 + 7 } = Max {17, 29, 27} = 29
6	Max {Time for Node 3 + Time for Activity E, Time for Node 5 + Time for Activity H} = Max {21 + 14 , 29 + 12} = Max {35, 41} = 41
7	Max {Time for Node 5 + Time for Activity I, Time for Node 6 + Time for Activity J} = Max {29 + 9, 41 + 10} = Max {38, 51} = 51
8	Time for Node 7 + Time for Activity J = 51+ 7 = 58

Earliest Start Times of the activities

Activity	Earliest Start Time (Weeks)
A	0
B	8

C	8
D	8
E	21
F	21
G	20
H	29
I	29
J	41
K	51

Backward pass

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 58
7	Time for Node 8 - Time for Activity K = 58 -7 = 51
6	Time for Node 7 - Time for Activity J = 51 -10 = 41
5	Min {Time for Node 6 - Time for Activity H, Time for Node 7 - Time for Activity I} = Min {41 - 12, 51 - 9} = Min {29, 42} = 29
4	Time for Node 5 - Time for Activity G = 29 - 7 = 22
3	Min {Time for Node 5 - Time for Activity F, Time for Node 6 - Time for Activity E} = Min {29 - 8, 41 - 14} = Min {21, 27} = 21

2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity D, = Min {21 - 13, 22 - 12, 29 - 9} = Min {8, 10, 20} = 8
1	Time for Node 2 - Time for Activity A = 8 - 8 = 0

Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
K	58
J	51
I	51
H	41
G	29
F	29
E	41
D	22
C	29
B	21
A	8

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	8	0	8	0	8	0
B	13	8	21	8	21	0
C	9	8	17	20	29	12
D	12	8	20	10	22	2
E	14	21	35	27	41	6
F	8	21	29	21	29	0
G	7	20	27	22	29	2
H	12	29	41	29	41	0
I	9	29	38	42	51	13
J	10	41	51	41	51	0
K	7	51	58	51	58	0

The activities with total float = 0 are A, B, F, H, J and K. They are the critical activities.
Project completion time = 58 weeks.

Calculation of slacks of the events

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

Event (Node)	Earliest Expected Time of Occurrence of Event	Latest Allowable Time of Occurrence of Event	Slack of the Event
1	0	0	0
2	8	8	0
3	21	21	0
4	20	22	2
5	29	29	0
6	41	41	0
7	51	51	0
8	58	58	0

Interpretation

On the basis of the slacks of the events, it is concluded that the occurrence of event 4 may be delayed upto a maximum period of 2 weeks while no other event cannot be delayed.

Questions

1. Explain the terms: The earliest and latest times of the activities of a project.
2. Explain the procedure to find the earliest expected time of an event.
3. Explain the procedure to find the latest allowable time of an event.
4. What is meant by the slack of an activity? How will you determine it?
5. Consider the project with the following details:

activity	Duration (weeks)
1→2	1
2 →3	3
2→4	7
3→4	5
3→5	8
4→5	4
5→6	1

Determine the earliest and the latest times of the activities. Calculate the total float for each activity and the slacks of the events.

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Lesson 7 - Crashing Of A Project

Lesson Outline

- The Idea Of Crashing Of A Project
- The Criterion Of Selection Of An Activity For Crashing
- Numerical Problems

Learning Objectives

After reading this lesson you should be able to

- understand the concept of crashing of a project
- choose an activity for crashing
- work out numerical problems

The Meaning Of Crashing

The process of shortening the time to complete a project is called crashing and is usually achieved by putting into service additional labour or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.

Steps In Project Crashing

Assumption: It is assumed that there is a linear relationship between time and cost.

Let us consider project crashing by the critical path method. The following four-step procedure is adopted.

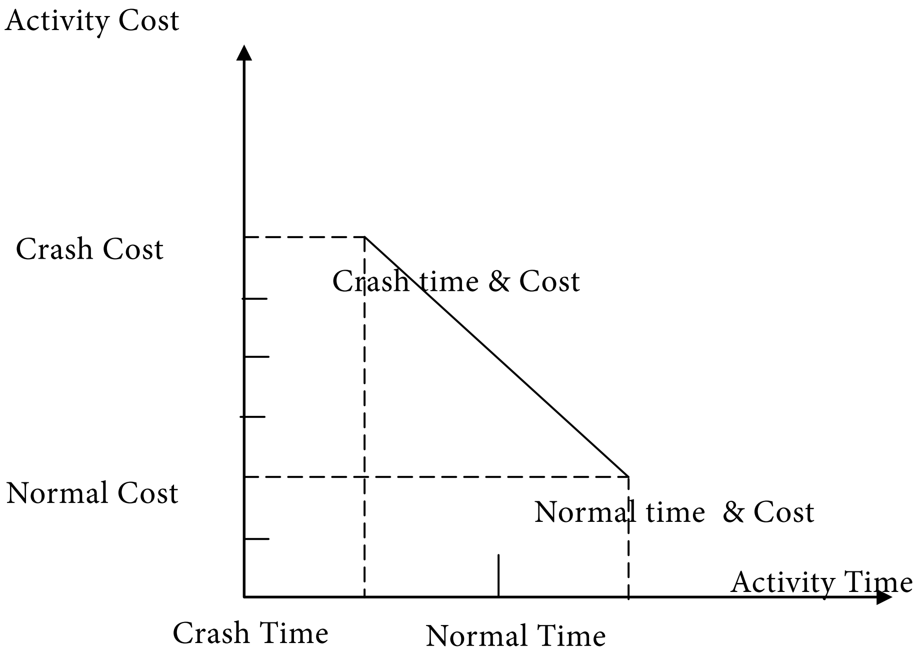
Step 1

Find the critical path with the normal times and normal costs for the activities and identify the critical activities.

Step 2

Find out the crash cost per unit time for each activity in the network. This is calculated by means of the following formula.

$$\text{Crash cost} / \text{Time period} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$



Step 3

Select an activity for crashing. The criteria for the selection is as follows:

Select the activity on the critical path with the smallest crash cost per unit time. Crash this activity to the maximum units of time as may be permissible by the given data.

Crashing an activity requires extra amount to be spent. However, even if the company is prepared to spend extra money, the activity time cannot be reduced beyond a certain limit in view of several other factors.

In step 1, we have to note that reducing the time of on activity along the critical path alone will reduce the completion time of a project. Because of this reason, we select an activity along the critical path for crashing.
In step 3, we have to consider the following question:

If we want to reduce the project completion time by one unit, which critical activity will involve the least additional cost?

On the basis of the least additional cost, a critical activity is chosen for crashing. If there is a tie between two critical activities, the tie can be resolved arbitrarily.

Step 4

After crashing an activity, find out which is the critical path with the changed conditions. Sometimes, a reduction in the time of an activity in the critical path may cause a non-critical path to become critical. If the critical path with which we started is still the longest path, then go to Step 3. Otherwise, determine the new critical path and then go to Step 3.

Problem 1

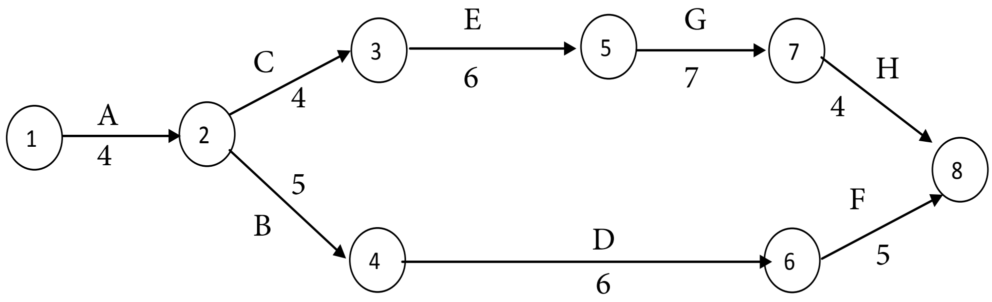
A project has activities with the following normal and crash times and cost: .

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	4	3	8,000	9,000
B	A	5	3	16,000	20,000
C	A	4	3	12,000	13,000
D	B	6	5	34,000	35,000
E	C	6	4	42,000	44,000
F	D	5	4	16,000	16,500
G	E	7	4	66,000	72,000
H	G	4	3	2,000	5,000

Determine a crashing scheme for the above project so that the total project time is reduced by 3 weeks.

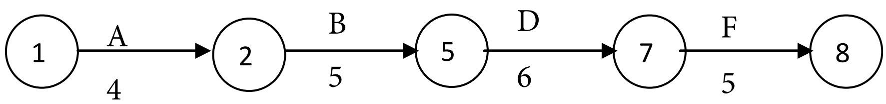
Solution

We have the following network diagram for the given project with normal costs:



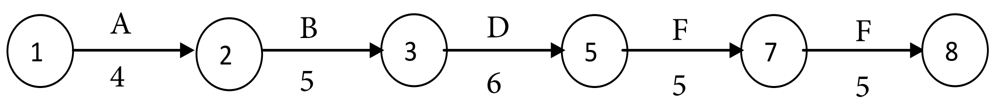
Beginning from the Start Node and terminating with the End Node, there are two paths for the network as detailed below:

Path I



The time for the path = 4 + 5 + 6 + 5 = 20 weeks.

Path II



The time for the path = 4 + 4 + 6 + 7 + 4 = 25 weeks.
Maximum of {20, 25} = 25.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. The non-critical activities are B, D and F.

Given that the normal time of activity A is 4 weeks while its crash time is 3 weeks. Hence the time of this activity can be reduced by one week if the management is prepared to spend an additional amount. However, the time cannot be reduced by more than one week even if the management may be prepared to spend more money. The normal cost of this activity is Rs. 8,000 whereas the crash cost is Rs. 9,000. From this, we see that crashing of activity A by one week will cost the management an extra amount of Rs. 1,000. In a similar fashion, we can work out the crash cost per unit time for the other activities also. The results are provided in the following table.

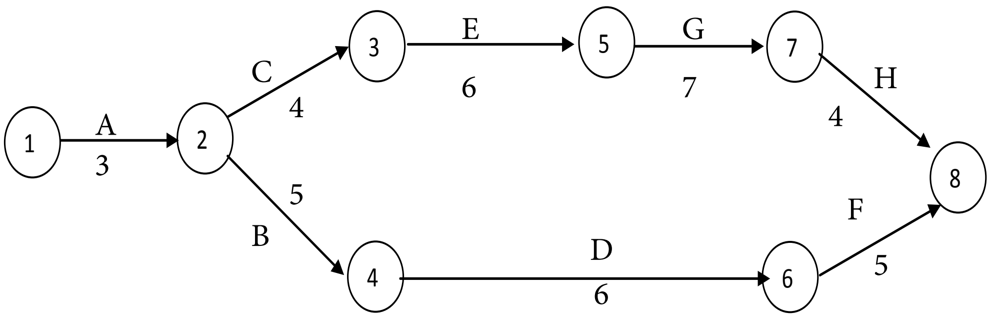
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	4	3	8,000	9,000	1,000	1	1,000
B	5	3	16,000	20,000	4,000	2	2,000
C	4	3	12,000	13,000	1,000	1	1,000
D	6	5	34,000	35,000	1,000	1	1,000
E	6	4	42,000	44,000	2,000	2	1,000
F	5	4	16,000	16,500	500	1	500
G	7	4	66,000	72,000	6,000	1	6,000
H	4	3	2,000	5,000	3,000	1	3,000

A non-critical activity can be delayed without delaying the execution of the whole project. But, if a critical activity is delayed, it will delay the whole project. Because of this reason, we have to select a critical activity for crashing. Here we have to choose one of the activities A, C, E, G and H The crash cost per unit time works out as follows:

Rs. 1,000 for A; Rs. 1,000 for C; Rs. 1,000 for E; Rs. 6,000 for G; Rs. 3,000 for H.

The maximum among them is Rs. 1,000. So we have to choose an activity with Rs.1,000 as the crash cost per unit time. However, there is a tie among A, C and E. The tie can be resolved arbitrarily. Let us select A for crashing. We reduce the time of A by one week by spending an extra amount of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



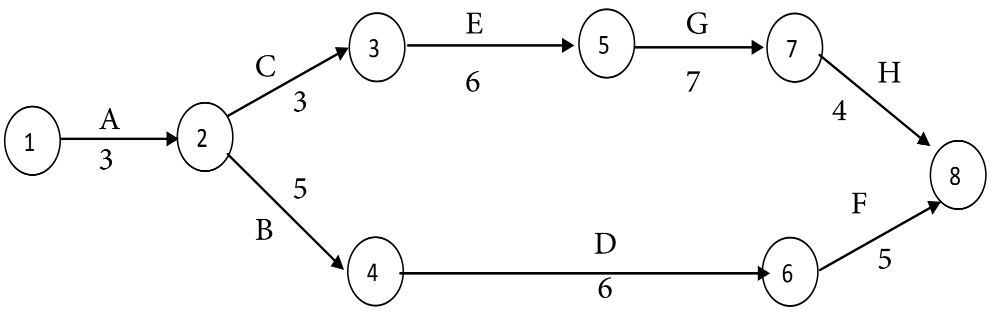
The revised time for Path I = 3 + 5 + 6 + 5 = 19 weeks.

The time for Path II = 3 + 4 + 6 + 7 + 4 = 24 weeks.

Maximum of {19, 24} = 24.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. However, the time for A cannot be reduced further. Therefore, we have to consider C, E, G and H for crashing. Among them, C and E have the least crash cost per unit time. The tie between C and E can be resolved arbitrarily. Suppose we reduce the time of C by one week with an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The time for Path I = 3 + 5 + 6 + 5 = 19 weeks.

The time for Path II = 3 + 3 + 6 + 7 + 4 = 23 weeks.

Maximum of {19, 23} = 23.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. Now the time for A or C cannot be reduced further. Therefore, we have to consider E, G and H for crashing. Among them, E has the least crash cost per unit time. Hence we reduce the time of E by one week with an extra cost of Rs. 1,000.

By the given condition, we have to reduce the project time by 3 weeks. Since this has been accomplished, we stop with this step.

Result: We have arrived at the following crashing scheme for the given project:

Reduce the time of A, C and E by one week each.

Project time after crashing is 22 weeks.

Extra amount required = 1,000 + 1,000 + 1,000 = Rs. 3,000.

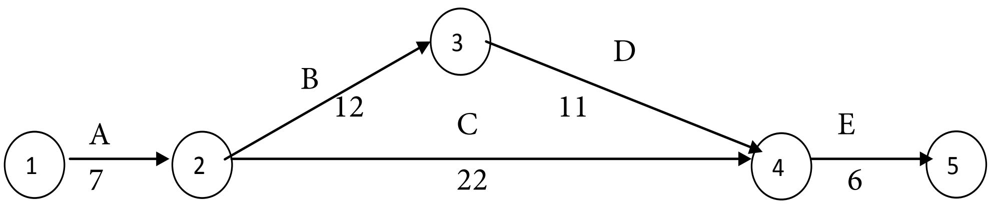
Problem 2

The management of a company is interested in crashing of the following project by spending an additional amount not exceeding Rs. 2,000. Suggest how this can be accomplished.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	7	6	15,000	18,000
B	A	12	9	11,000	14,000
C	A	22	21	18,500	19,000
D	B	11	10	8,000	9,000
E	C, D	6	5	4,000	4,500

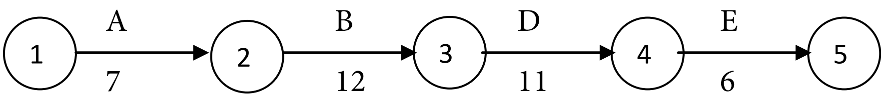
Solution

We have the following network diagram for the given project with normal costs:



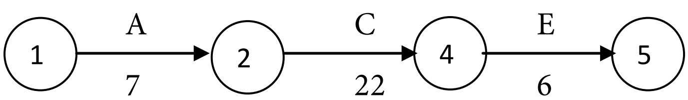
There are two paths for this project as detailed below:

Path I



The time for the path = 7 + 12 + 11 + 6 = 36 weeks.

Path II



The time for the path = 7 + 22 + 6 = 35 weeks.

Maximum of {36, 35} = 36.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.

The crash cost per unit time for the activities in the project are provided in the following table.

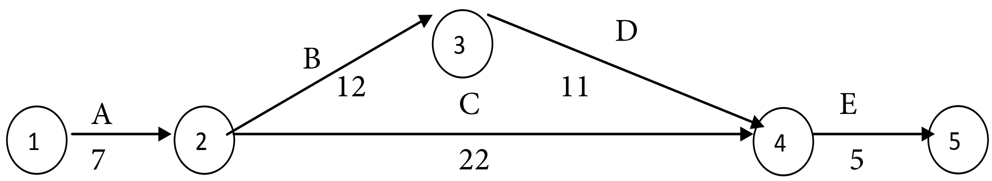
Activity	Normal	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	7	6	15,000	18,000	3,000	1	3,000
B	12	9	11,000	14,000	3,000	3	1,000
C	22	21	18,500	19,000	500	1	500
D	11	10	8,000	9,000	1,000	1	1,000
E	6	5	4,000	4,500	500	1	500

We have to choose one of the activities A, B, D and E for crashing. The crash cost per unit time is as follows:

Rs. 3,000 for A; Rs. 1,000 for B; Rs. 1,000 for D; Rs. 500 for E.

The least among them is Rs. 500. So we have to choose the activity E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:

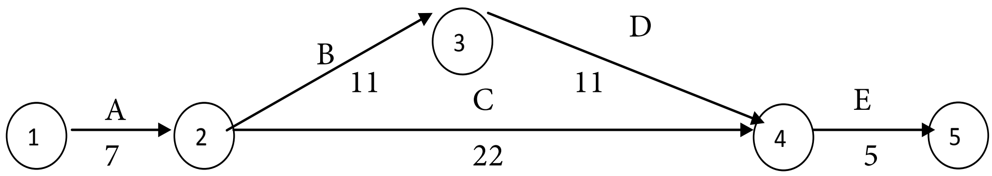


The revised time for Path I = 7 + 12 + 11 + 5 = 35 weeks.
The time for Path II = 7 + 22 + 5 = 34 weeks.
Maximum of {35, 34} = 35.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.

The time of E cannot be reduced further. So we cannot select it for crashing. Next B and have the smallest crash cost per unit time. Let us select B for crashing. Let us reduce the time of E by one week at an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 7 + 11 + 11 + 5 = 34 weeks.

The time for Path II = 7 + 22 + 5 = 34 weeks.

Maximum of {34, 34} = 34.

Since both paths have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time. In path I, the activities are A, B, D and E. In path II, the activities are A, C and E.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this state has been met, we stop with this step.

Result: The following crashing scheme is suggested for the given project:

Reduce the time of E, B and C by one week each.

Project time after crashing is 33 weeks.

Extra amount required = 500 + 1,000 + 500 = Rs. 2,000.

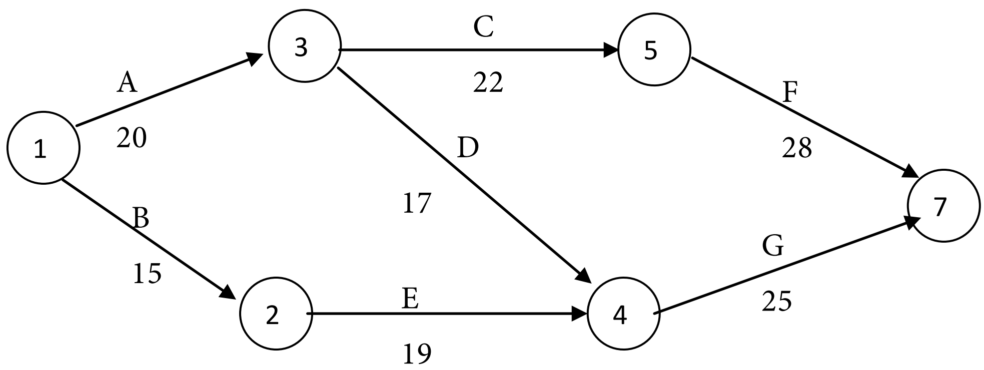
Problem 3

The manager of a company wants to apply crashing for the following project by spending an additional amount not exceeding Rs. 2,000. Offer your suggestion to the manager.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	20	19	8,000	10,000
B	-	15	14	16,000	19,000
C	A	22	20	13,000	14,000
D	A	17	15	7,500	9,000
E	B	19	18	4,000	5,000
F	C	28	27	3,000	4,000
G	D, E	25	24	12,000	13,000

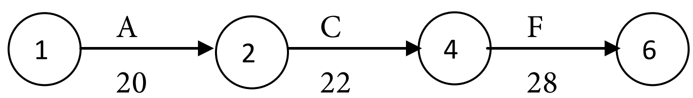
Solution

We have the following network diagram for the given project with normal costs:



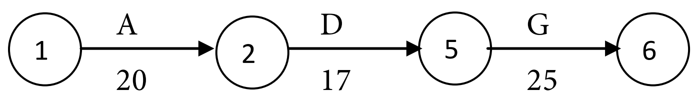
There are three paths for this project as detailed below:

Path I



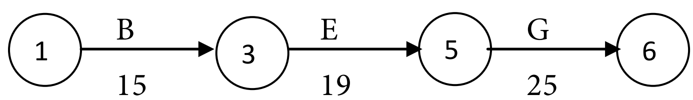
The time for the path = 20 + 22 + 28= 70 weeks.

Path II



The time for the path = 20 + 17 + 25= 62 weeks.

Path III



The time for the path = 15+19 +25 = 69 weeks.
Maximum of {70, 62, 69} = 70.

Therefore Path I is the critical path and the critical activities are A, C and F. The non-critical activities are B, D, E and G.

The crash cost per unit time for the activities in the project are provided in the following table

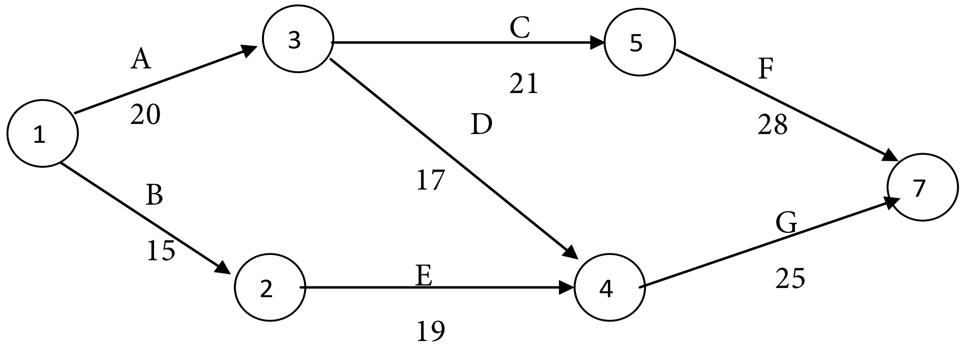
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	20	19	8,000	10,000	2,000	1	2,000
B	15	14	16,000	19,000	3,000	1	3,000
C	22	20	13,000	14,000	1,000	2	500
D	17	15	7,500	9,000	1,500	2	750
E	19	18	4,000	5,000	1,000	1	1,000
F	28	27	3,000	4,000	1,000	1	1,000
G	25	24	12,000	13,000	1,000	1	1,000

We have to choose one of the activities A, C and F for crashing. The crash cost per unit time is as follows:

Rs. 2,000 for A; Rs. 500 for C; Rs. 1,000 for F.

The least among them is Rs. 500. So we have to choose the activity C for crashing. We reduce the time of C by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 20 + 21 + 28= 69 weeks.

The time for Path II = 20 + 17 + 25= 62 weeks.

The time for Path III = 15+19 +25 = 69 weeks.

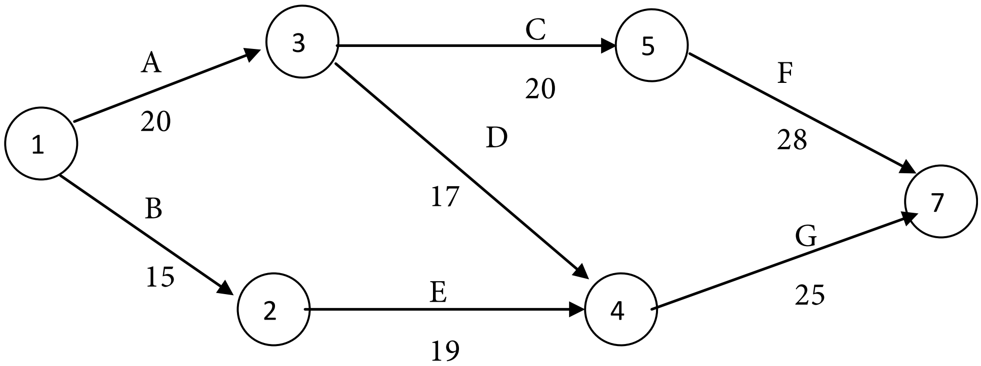
Maximum of {69, 62, 69} = 69.

Since paths I and III have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time.

In path I, the activities are A, C and F. In path III, the activities are B, E and G.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 20 + 20 + 28= 68 weeks.

The time for Path II = 20 + 17 + 25= 62 weeks.

The time for Path III = 15+19 +25 = 69 weeks.

Maximum of {68, 62, 69} = 69.

Therefore path III is the critical activities. Hence we have to select an activity from Path III for crashing. We see that the crash cost per unit time is as follows:

Rs. 3,000 for B; Rs. 1,000 for E; Rs. 1,000 for G.

The least among them is Rs. 1,000. So we can select either E or G for crashing. Let us select E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 1,000.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this condition has been reached, we stop with this step.

Result: The following crashing scheme is suggested for the given project:
Reduce the time of C by 2 weeks and that of E by one week.

Project time after crashing is 67 weeks.
Extra amount required = 2 x 500 + 1,000 = Rs. 2,000.

Questions

1. Explain the concept of crashing of a project.
2. Explain the criterion for selection of an activity for crashing.

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UNIT V

Game Theory, Goal Programming & Queuing Theory
Lesson 1 - Basic Concepts In Game Theory

Lesson Outline

- Introduction To The Theory Of Games
- The Definition Of A Game
- Competitive Game
- Managerial Applications Of The Theory Of Games
- Key Concepts In The Theory Of Games
- Types Of Games

Learning Objectives

After reading this lesson you should be able to

- Understand the concept of a game
- Grasp the assumptions in the theory of games
- Appreciate the managerial applications of the theory of games
- Understand the key concepts in the theory of games
- Distinguish between different types of games

Introduction To Game Theory

Game theory seeks to analyse competing situations which arise out of conflicts of interest. Abraham Maslow's hierarchical model of human needs lays emphasis on fulfilling the basic needs such as food, water, clothes, shelter, air, safety and security. There is conflict of interest between animals and plants in the consumption of natural resources. Animals compete among themselves for securing food. Man competes with animals to earn his food. A man also competes with another man. In the past, nations waged wars to expand the territory of their rule. In the present day world, business organizations compete with each other in getting the market share. The conflicts of interests of human beings are not confined to the basic needs alone. Again considering Abraham Maslow's model of human needs, one can realize that conflicts also arise due to the higher levels of human needs such as love, affection, affiliation, recognition, status, dominance, power, esteem, ego, self-respect, etc. Sometimes one witnesses clashes of ideas of intellectuals also. Every intelligent and rational participant in a conflict wants to be a winner but not all participants can be the winners at a time. The situations of conflict gave birth to Darwin's theory of the 'survival of the fittest'. Nowadays the concepts of conciliation, co-existence, co-operation, coalition and consensus are gaining ground. Game theory is another tool to examine situations of conflict so as to identify the courses of action to be followed and to take appropriate decisions in the long run. Thus this theory assumes importance from managerial perspectives. The pioneering work on the theory of games was done by von Neumann and Morgenstern through their publication entitled 'The Theory of Games and Economic Behaviour' and subsequently the subject was developed by several experts. This theory can offer valuable guidelines to a manager in 'strategic management' which can be used in the decision making process for merger, take-over, joint venture, etc. The results obtained by the application of this theory can serve as an early warning to the top level management in meeting the threats from the competing business organizations and for the conversion of the internal weaknesses and external threats into opportunities and strengths, thereby achieving the goal of maximization of profits. While this theory does not describe any procedure to play a game, it will enable a participant to select the appropriate strategies to be followed in the pursuit of his goals. The situation of failure in a game would activate a participant in the analysis of the relevance of the existing strategies and lead him to identify better, novel strategies for the future occasions.

Definitions of game theory

There are several definitions of game theory. A few standard definitions are presented below.

In the perception of Robert Mockler, "Game theory is a mathematical technique helpful in making decisions in situations of conflicts, where the success of one part depends at the expense of others, and where the individual decision maker is not in complete control of the factors influencing the outcome".

The definition given by William G. Nelson runs as follows: "Game theory, more properly **the theory of games of strategy,** is a mathematical method of analyzing a conflict. The alternative is not between this decision or that decision, but between this strategy or that strategy to be used against the conflicting interest".

In the opinion of Matrin Shubik, "Game theory is a method of the study of decision making in situation of conflict. It deals with human processes in which the individual decision-unit is not in complete control of other decision-units entering into the environment".

According to von Neumann and Morgenstern, "The 'Game' is simply the totality of the rules which describe it. Every particular instance at which the game is played – in a particular way – from beginning to end is a 'play'. The game consists of a sequence of moves, and the play of a sequence of choices".

J.C.C McKinsey points out a valid distinction between two words, namely 'game' and 'play'. According to him, "game refers to a particular realization of the rules".

In the words of O.T. Bartos, "The theory of games can be used for 'prescribing' how an intelligent person should go about resolving social conflicts, ranging all the way from open warfare between nations to disagreements between husband and wife".

Martin K Starr gave the following definition: "Management models in the competitive sphere are usually termed game models. By studying game theory, we can obtain substantial information into management's role under competitive conditions, even though much of the game theory is neither directly operational nor implementable".

According to Edwin Mansfield, "A game is a competitive situation where two or more persons pursue their own interests and no person can dictate the outcome. Each player, an entity with the same interests, make his own decisions. A player can be an individual or a group".

Assumptions for a Competitive Game

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold:

The number of competitors is finite, say N.

A finite set of possible courses of action is available to each of the N competitors.

A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that al the players select their courses of action simultaneously. As a result, no competitor will be in a position to know the choices of his competitors.

The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1. Analysis of the market strategies of a business organization in the long run.
2. Evaluation of the responses of the consumers to a new product.
3. Resolving the conflict between two groups in a business organization.
4. Decision making on the techniques to increase market share.
5. Material procurement process.
6. Decision making for transportation problem.
7. Evaluation of the distribution system.
8. Evaluation of the location of the facilities.
9. Examination of new business ventures and
10. Competitive economic environment.

Key concepts in the Theory of Games

Several of the key concepts used in the theory of games are described below:

Players

The competitors or decision makers in a game are called the players of the game.

Strategies

The alternative courses of action available to a player are referred to as his strategies.

Pay off

The outcome of playing a game is called the pay off to the concerned player.

Optimal Strategy

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game

A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

Non-zero sum game

Games with "less than complete conflict of interest" are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

Mixed strategy

If there is no one specific strategy as the 'best strategy' for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game

A game in which N-players take part is called an N-person game.

Maximin-Minimax Principle

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

Negotiable or cooperative game

If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

Non-negotiable or non-cooperative game

If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

Saddle point

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called the value of the game and the corresponding strategies are called the pure strategies.

Dominance

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Types of Games

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

Games of perfect information and games of imperfect information

A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

Games with finite number of moves / players and games with unlimited number of moves

A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

Constant-sum games

If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero-sum game.

2x2 two person game and 2xn and mx2 games

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game.

A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game.

Two-person zero sum games are said to be larger if each of the two players has 3 or more choices.

The examination of 3x3 and larger games is involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

Non-constant games

Consider a game with two players. If the sum of the payoffs to the two players is not constant in all the plays of the game, then we call it a non-constant game.

Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

Questions

1. Explain the concept of a game.
2. Define a game.
3. State the assumptions for a competitive game.
4. State the managerial applications of the theory of games.
5. Explain the following terms: strategy, pay-off matrix, saddle point, pure strategy and mixed strategy.
6. Explain the following terms: two person game, two person zero sum game, value of a game, 2xn game and mx2 game.

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Lesson 2 - Two-Person Zero Sum Games

Lesson Outline

- The Concept Of A Two-Person Zero Sum Game
- The Assumptions For A Two-Person Zero Sum Game
- Minimax And Maximin Principles

Learning Objectives

After reading this lesson you should be able to

- Understand the concept of a two-person zero sum game
- Have an idea of the assumptions for a two-person zero sum game
- Understand Minimax and Maximin principles
- Solve a two-person zero sum game
- Interpret the results from the payoff matrix of a two-person zero sum game

Definition of two-person zero sum game

A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

Payoff matrix

When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix.

Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies. Consider the following payoff matrix.

Player B's strategies

Player A's strategies

$$\begin{array}{c} \\ A_1 \\ A_1 \\ \vdots \\ Am \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Player A wishes to gain as large a payoff a_{ij} as possible while player B will do his best to reach as small a value a_{ij} as possible where the gain to player B and loss to player A be (- a_{ij}).

Assumptions for two-person zero sum game

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.

a. Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.
b. Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.
c. The decisions of both players are made individually prior to the play with no communication between them.
d. The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
e. Both players know the possible payoffs of themselves and their opponents.

Minimax and Maximin Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games.

The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as minimax-maximin principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his i[th] strategy, then he gains at least the payoff min a_{ij}, which is minimum of the i[th] row elements in the payoff matrix. Since his objective is to maximize his payoff, he can choose strategy *i* so as to make his payoff as large as possible. i.e., a payoff which is not less than $\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij}$.

Similarly player B can choose jth column elements so as to make his loss not greater than $\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij}$.

If the maximin value for a player is equal to the minimax value for another player, i.e.

$$\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij} = V = \min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij}$$

then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal.

The amount of payoff, i.e., V at an equilibrium point is known as the ***value of the game.***

The optimal strategies can be identified by the players in the long run.

Fair game

The game is said to be fair if the value of the game V = 0.

Problem 1

Solve the game with the following pay-off matrix.

Player B

Strategies

Player A Strategies		*I*	*II*	*III*	*IV*	*V*
	1	–2	5	–3	6	7
	2	4	6	8	–1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

Solution

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of {-3, -1, 2, 12} = 12

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of {15, 14, 18, 12, 20}= 12

We see that the maximum of row minima = the minimum of the column maxima. So the game has a saddle point. The common value is 12. Therefore the value V of the game = 12.

Interpretation

In the long run, the following best strategies will be identified by the two players:

The best strategy for player A is strategy 4.

The best strategy for player B is strategy IV.

The game is favourable to player A.

Problem 2

Solve the game with the following pay-off matrix

Player Y

Strategies

		I	*II*	*III*	*IV*	*V*
Player X Strategies	1	9	12	7	14	26
	2	25	35	20	28	30
	3	7	6	−8	3	2
	4	8	11	13	−2	1

Solution

First consider the minimum of each row.

Row	**Minimum Value**
1	7
2	20
3	-8
4	-2

Maximum of {7, 20, −8, -2} = 20

Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

Minimum of {25, 35, 20, 28, 30}= 20

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20. This indicates that the value V of the game is 20.

Interpretation.

The best strategy for player X is strategy 2.

The best strategy for player Y is strategy III.

The game is favourable to player A.

Problem 3

Solve the following game:

Player B

Strategies

Player A Strategies		*I*	*II*	*III*	*IV*
	1	1	–6	8	4
	2	3	–7	2	–8
	3	5	–5	–1	0
	4	3	–4	5	7

Solution

First consider the minimum of each row.

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

Maximum of {-6, -8, -5, -4} = -4

Next consider the maximum of each column.

Column	Maximum Value
1	5
2	-4
3	8
4	7

Minimum of {5, -4, 8, 7}= - 4

Since the max {row minima} = min {column maxima}, the game under consideration has a saddle point. The common value is –4. Hence the value of the game is –4.

Interpretation

The best strategy for player A is strategy 4.

The best strategy for player B is strategy II.

Since the value of the game is negative, it is concluded that the game is favourable to player B.

Questions

1. What is meant by a two-person zero sum game? Explain.
2. State the assumptions for a two-person zero sum game.
3. Explain Minimax and Maximin principles.
4. How will you interpret the results from the payoff matrix of a two-person zero sum game? Explain.
5. What is a fair game? Explain.
6. Solve the game with the following pay-off matrix.

Player B

Strategies

Player A Strategies		I	II	III	IV	V
	1	7	5	2	3	9
	2	10	8	7	4	5
	3	9	12	0	2	1
	4	11	−2	−1	3	4

Answer: Best strategy for A: 2

Best strategy for B: IV

V = 4

The game is favourable to player A

7. Solve the game with the following pay-off matrix.

Player B

Strategies

Player A Strategies		I	II	III	IV	V
	1	−2	−3	8	7	0
	2	1	−7	−5	−2	3
	3	4	−2	3	5	−1
	4	6	−4	5	4	7

Answer: Best strategy for A: 3

Best strategy for B: II

V = -2

The game is favourable to player B

8. Solve the game with the following pay-off matrix.

Player B

Player A

6	4
7	5

Answer: Best strategy for A: 2

Best strategy for B: II

V = 5

The game is favourable to player A

9. Solve the following game and interpret the result.

Player B

Strategies

Player A Strategies

	I	*II*	*III*	*IV*
1	–3	–7	1	3
2	–1	2	– 3	1
3	0	4	2	6
4	–2	–1	– 5	1

Answer: Best strategy for A: 3

Best strategy for B: I

V = 0

The value V = 0 indicates that the game is a fair one.

10. Solve the following game:

Player B

Strategies

Player A Strategies

	I	*II*	*III*
1	1	8	2
2	3	5	6
3	2	2	1

Answer: Best strategy for A: 2

Best strategy for B: I

V = 3

The game is favourable to player A

11. Solve the game

Player B

Player A

	I	II	III	IV
1	4	–1	2	0
2	–3	–5	–9	– 2
3	2	–8	0	–11

Answer : V = -1

12. Solve the game

Player Y

Player X

	I	II	III	IV	V
1	4	0	1	7	–1
2	0	–3	–5	–7	5
3	3	2	3	4	3
4	–6	4	–1	0	5
5	0	0	6	0	0

Answer : V = 2

13. Solve the game

Player B

Player A

	I	II	III	IV	V
1	9	3	4	4	2
2	8	6	8	5	12
3	10	7	19	18	14
4	8	6	8	11	6
5	3	5	16	10	8

Answer : V = 7

14. Solve the game

Player Y

Player X

17	10	12	5	4	8
2	5	6	7	6	9
7	6	9	2	3	1
10	11	14	8	13	8
20	18	17	10	15	17
12	11	15	9	5	11

Answer : V = 10

15. Solve the game

Player B

Player A

12	14	8	7	4	9
2	13	6	7	9	8
13	6	8	6	3	1
14	9	10	8	9	6
20	18	17	11	14	16
8	12	16	9	6	13

Answer : V = 11

16. Examine whether the following game is fair.

Player Y

Player X

6	–4	–3	–2
3	5	0	8
7	–2	–6	5

Answer : V = 0.

Therefore, it is a fair game.

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Lesson 3 - Games With No Saddle Point

Lesson Outline

- The Concept Of A 2X2 Game With No Saddle Point
- The Method Of Solution

Learning Objectives

After reading this lesson you should be able to

- Understand the concept of a 2x2 game with no saddle point
- Know the method of solution of a 2x2 game without saddle point
- Solve a game with a given payoff matrix
- Interpret the results obtained from the payoff matrix

2 x 2 zero-sum game

When each one of the first player A and the second player B has exactly two strategies, we have a 2 x 2 game.

Motivating point

First let us consider an illustrative example

Problem 1

Examine whether the following 2 x 2 game has a saddle point

Player B

Player A

3	5
4	2

Solution

First consider the minimum of each row.

Row	Minimum Value
1	3
2	2

Maximum of {3, 2} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5}= 4

We see that max {row minima} and min {column maxima} are not equal. Hence the game has no saddle point.

Method of solution of a 2x2 zero-sum game without saddle point

Suppose that a 2x2 game has no saddle point. Suppose the game has the following pay-off matrix.

Player B

$$\text{Player A} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since this game has no saddle point, the following condition shall hold:

$$Max\{Min\{a,b\}, Min\{c,d\}\} \neq Min\{Max\{a,c\}, Max\{b,d\}\}$$

In this case, the game is called a mixed game. ***No strategy of Player A can be called the best strategy for him. Therefore A has to use both of his strategies. Similarly no strategy of Player B can be called the best strategy for him and he has to use both of his strategies.***

Let p be the probability that Player A will use his first strategy. Then the probability that Player A will use his second strategy is 1-p.

<u>If Player B follows his first strategy</u>

Expected value of the pay-off to Player A

$$= \left\{ \begin{matrix} \text{Expected value of the pay-off to Player A} \\ \text{arising from his first strategy} \end{matrix} \right\} + \left\{ \begin{matrix} \text{Expected value of the pay-off to Player A} \\ \text{arising from his second strategy} \end{matrix} \right\}$$

$$= ap + c(1-p) \qquad (1)$$

In the above equation, note that the expected value is got as the product of the corresponding values of the pay-off and the probability.

<u>If Player B follows his second strategy</u>

$$\left. \begin{matrix} \text{Expected value of the} \\ \text{pay-off to Player A} \end{matrix} \right\} = bp + d(1-p) \qquad (2)$$

If the expected values in equations (1) and (2) are different, Player B will prefer the minimum of the two expected values that he has to give to player A. Thus B will have a pure strategy. This contradicts our assumption that the game is a mixed one. Therefore the expected values of the pay-offs to Player A in equations (1) and (2) should be equal. Thus we have the condition

$$\begin{aligned}
ap + c(1-p) &= bp + d(1-p) \\
ap - bp &= (1-p)[d-c] \\
p(a-b) &= (d-c) - p(d-c) \\
p(a-b) + p(d-c) &= d-c \\
p(a-b+d-c) &= d-c \\
p &= \frac{d-c}{(a+d)-(b+c)} \\
1-p &= \frac{a+d-b-c-d+c}{(a+d)-(b+c)} \\
&= \frac{a-b}{(a+d)-(b+c)}
\end{aligned}$$

$$\left\{\begin{matrix}\text{The number of times A} \\ \text{will use first strategy}\end{matrix}\right\} : \left\{\begin{matrix}\text{The number of times A} \\ \text{will use second strategy}\end{matrix}\right\} = \frac{d-c}{(a+d)-(b+c)} : \frac{a-b}{(a+d)-(b+c)}$$

The expected pay-off to Player A

$$\begin{aligned}
&= ap + c(1-p) \\
&= c + p(a-c) \\
&= c + \frac{(d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{c\{(a+d)-(b+c)\} + (d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{ac + cd - bc - c^2 + ad - cd - ac + c^2)}{(a+d)-(b+c)} \\
&= \frac{ad-bc}{(a+d)-(b+c)}
\end{aligned}$$

Therefore, the value V of the game is

$$\frac{ad-bc}{(a+d)-(b+c)}$$

To find the number of times that B will use his first strategy and second strategy:

Let the probability that B will use his first strategy be r. Then the probability that B will use his second strategy is 1-r.

<u>When A use his first strategy</u>

The expected value of loss to Player B with his first strategy = ar

The expected value of loss to Player B with his second strategy = b(1-r)

Therefore the expected value of loss to B = ar + b(1-r) (3)

<u>When A use his second strategy</u>

The expected value of loss to Player B with his first strategy = cr

The expected value of loss to Player B with his second strategy = d(1-r)

Therefore the expected value of loss to B = cr + d(1-r) (4)

If the two expected values are different then it results in a pure game, which is a contradiction. Therefore the expected values of loss to Player B in equations (3) and (4) should be equal. Hence we have the condition

$$
\begin{aligned}
& ar + b(1-r) = cr + d(1-r) \\
& ar + b - br = cr + d - dr \\
& ar - br - cr + dr = d - b \\
& r(a - b - c + d) = d - b \\
& r = \frac{d-b}{a-b-c+d} \\
& \quad = \frac{d-b}{(a+d)-(b+c)}
\end{aligned}
$$

Problem 2

Solve the following game

$$
\begin{array}{cc}
 & \mathbf{Y} \\
X & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}
\end{array}
$$

Solution

First consider the row minima.

Row	**Minimum Value**
1	2
2	1

Maximum of {2, 1} = 2

Next consider the maximum of each column

Column	**Maximum Value**
1	4
2	5

Minimum of {4, 5}= 4

We see that

Max {row minima} $\neq$ min {column maxima}

So the game has no saddle point. Therefore it is a mixed game.

We have a = 2, b = 5, c = 4 and d = 1.

Let p be the probability that player X will use his first strategy. We have

$$p = \frac{d-c}{(a+d)-(b+c)}$$

$$= \frac{1-4}{(2+1)-(5+4)}$$

$$= \frac{-3}{3-9}$$

$$= \frac{-3}{-6}$$

$$= \frac{1}{2}$$

The probability that player X will use his second strategy is

$1\text{-p} = 1\text{-} \frac{1}{2} = \frac{1}{2}$.

Value of the game V = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{2-20}{3-9} = \frac{-18}{-6} = 3$.

Let r be the probability that Player Y will use his first strategy. Then the probability that Y will use his second strategy is (1-r). We have

$$
\begin{aligned}
r &= \frac{d-b}{(a+d)-(b+c)} \\
&= \frac{1-5}{(2+1)-(5+4)} \\
&= \frac{-4}{3-9} \\
&= \frac{-4}{-6} \\
&= \frac{2}{3} \\
1-r &= 1-\frac{2}{3} = \frac{1}{3}
\end{aligned}
$$

Interpretation

$$p : (1\text{-p}) = \frac{1}{2} : \frac{1}{2}$$

Therefore, out of 2 trials, player X will use his first strategy once and his second strategy once.

$$r : (1\text{-r}) = \frac{2}{3} : \frac{1}{3}$$

Therefore, out of 3 trials, player Y will use his first strategy twice and his second strategy once.

Questions

1. What is a 2x2 game with no saddle point? Explain.
2. Explain the method of solution of a 2x2 game without saddle point.
3. Solve the following game

$$\begin{array}{cc} & \textbf{Y} \\ X & \begin{bmatrix} 12 & 4 \\ 3 & 7 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{4}$, $V = 6$

4. Solve the following game

$$\begin{array}{cc} & \textbf{Y} \\ X & \begin{bmatrix} 5 & -4 \\ -9 & 3 \end{bmatrix} \end{array}$$

Answer: $p = \frac{4}{7}$, $r = \frac{1}{3}$, $V = -1$

5. Solve the following game

$$\begin{array}{cc} & \textbf{Y} \\ X & \begin{bmatrix} 10 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{4}$, $r = \frac{1}{2}$, $V = 7$

6. Solve the following game

$$\begin{array}{cc} & \textbf{Y} \\ X & \begin{bmatrix} 20 & 8 \\ -2 & 10 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{2}$, $r = \frac{1}{12}$, $V = 9$

7. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} 10 & 2 \\ 1 & 5 \end{bmatrix}}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{4}$, $V = 4$

8. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} 12 & 6 \\ 6 & 9 \end{bmatrix}}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{3}$, $V = 8$

9. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}}$$

Answer: $p = \frac{1}{2}$, $r = \frac{1}{2}$, $V = 9$

10. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} 16 & 4 \\ 4 & 8 \end{bmatrix}}$$

Answer: $p = \frac{1}{4}$, $r = \frac{1}{4}$, V = 7

11. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} -11 & 5 \\ 7 & -9 \end{bmatrix}}$$

Answer: $p = \frac{1}{2}$, $r = \frac{7}{16}$, V = - 2

12. Solve the following game

$$X \quad \overset{\textbf{Y}}{\begin{bmatrix} -9 & 3 \\ 5 & -7 \end{bmatrix}}$$

Answer: $p = \frac{1}{2}$, $r = \frac{5}{12}$, V = - 2

****

Lesson 4 - The Principle Of Dominance

Lesson Outline

- The Principle Of Dominance
- Dividing A Game Into Sub Games

Learning Objectives

After reading this lesson you should be able to

- Understand the principle of dominance
- Solve a game using the principle of dominance
- Solve a game by dividing a game into sub games

The Principle Of Dominance

In the previous lesson, we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.

In a given pay-off matrix A, we say that the ith row dominates the kth row if

$$a_{ij} \geq a_{kj} \text{ for all } j = 1,2,\ldots,n$$

and

$$a_{ij} > a_{kj} \text{ for at least one } j.$$

In such a situation player A will never use the strategy corresponding to kth row, because he will gain less for choosing such a strategy.

Similarly, we say the pth column in the matrix dominates the qth column if

$$a_{ip} \le a_{iq} \text{ for all i = 1,2,…,m}$$

and

$$a_{ip} < a_{iq} \text{ for at least one i.}$$

In this case, the player B will loose more by choosing the strategy for the qth column than by choosing the strategy for the pth column. So he will never use the strategy corresponding to the qth column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

Problem 1

Solve the game with the following pay-off matrix

Player B

$$\text{Player A} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} I & II & III & IV \end{matrix} \\ \begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix} \end{array}$$

Solution

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3

Next consider the maximum of each column

Column	Maximum Value
1	6
2	4
3	4
4	7
4	6

Minimum of {6, 4, 7, 6}= 4

The following condition holds:

$$\text{Max \{row minima\}} \neq \text{min \{column maxima\}}$$

Therefore we see that there is no saddle point for the game under consideration.

Compare columns II and III

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now we have the reduced game

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} I & II & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4

The game reduces to the following:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} I & II \end{array} \\ \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{bmatrix} 3 & 4 \\ 6 & 3 \end{bmatrix}$$

Again, there is no saddle point. We have a 2x2 matrix. Take this matrix as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then we have a = 3, b = 4, c = 6 and d = 3. Use the formulae for p, 1-p, r, 1-r and V.

$$\begin{aligned} p &= \frac{d-c}{(a+d)-(b+c)} \\ &= \frac{3-6}{(3+3)-(6+4)} \\ &= \frac{-3}{6-10} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

$$1-p = 1-\frac{3}{4} = \frac{1}{4}$$

$$\begin{aligned} r &= \frac{d-b}{(a+d)-(b+c)} \\ &= \frac{3-4}{(3+3)-(6+4)} \\ &= \frac{-1}{6-10} \\ &= \frac{-1}{-4} \\ &= \frac{1}{4} \end{aligned}$$

$$1-r = 1-\frac{1}{4} = \frac{3}{4}$$

The value of the game

$$\begin{aligned} V &= \frac{ad-bc}{(a+d)-(b+c)} \\ &= \frac{3x3-4x6}{-4} \\ &= \frac{-15}{-4} \\ &= \frac{15}{4} \end{aligned}$$

Thus, X = $\left(\frac{3}{4},\frac{1}{4},0,0\right)$ and Y = $\left(\frac{1}{4},\frac{3}{4},0,0\right)$ are the optimal strategies.

Method of convex linear combination

A strategy, say s, can also be dominated if it is inferior to a convex linear combination of several other pure strategies. In this case if the domination is strict, then the strategy s can be deleted. If strategy s dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per the above rules. Let us consider an example to illustrate this case.

Problem 2

Solve the game with the following pay-off matrix for firm A

Firm B

$$\text{Firm A}\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \\ 4 & -1 & 5 & 7 & 3 \end{bmatrix} \end{array}$$

Solution

First consider the minimum of each row

Row	**Minimum Value**
1	-2
2	0
3	-6
4	-3
5	-1

Maximum of {-2, 0, -6, -3, -1} = 0

Next consider the maximum of each column.

Column	**Maximum Value**
1	4
2	8
3	6
4	8
5	6

Minimum of { 4, 8, 6, 8, 6}= 4

Hence,

Maximum of {row minima} $\neq$ minimum of {column maxima}

So we see that there is no saddle point. Compare the second row with the fifth row. Each element in the second row exceeds the corresponding element in the fifth row. Therefore, A_2 dominates A_5 . The choice is for firm A. It will retain strategy A_2 and give up strategy A_5. Therefore the game reduces to the following.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \end{bmatrix} \end{array}$$

Compare the second and fourth rows. We see that A_2 dominates A_4. So, firm A will retain the strategy A_2 and give up the strategy A_4 . Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \end{bmatrix} \end{array}$$

Compare the first and fifth columns. It is observed that B1 dominates B5. The choice is for firm B. It will retain the strategy B_1 and give up the strategy B_5 . Thus the game reduces to the following

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 \\ 4 & 0 & 6 & 8 \\ -2 & -6 & -4 & 4 \end{bmatrix} \end{array}$$

Compare the first and fourth columns. We notice that B1 dominates B4. So firm B will discard the strategy B_4 and retain the strategy B_1 . Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 \\ 4 & 0 & 6 \\ -2 & -6 & -4 \end{bmatrix} \end{array}$$

For this reduced game, we check that there is no saddle point.
Now none of the pure strategies of firms A and B is inferior to any of their other strategies. But, we observe that convex linear combination of the strategies B_2 and B_3 dominates B_1 , i.e. the averages of payoffs due to strategies B_2 and B_3 ,

$$\left\{\frac{8-2}{2}, \frac{0+6}{2}, \frac{-6-4}{2}\right\} = \{3, 3, -5\}$$

dominate B_1 . Thus B_1 may be omitted from consideration. So we have the reduced matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \\ -6 & -4 \end{bmatrix} \end{array}$$

Here, the average of the pay-offs due to strategies A_1 and A_2 of firm A,

i.e. $\left\{\frac{8+0}{2}, \frac{-2+6}{2}\right\} = \{4,2\}$ dominates the pay-off due to A_3 . So we get a

new reduced 2x2 pay-off matrix.

Firm B's strategy

Firm A's strategy $\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \end{bmatrix} \end{array}$

We have a = 8, b = -2, c = 0 and d = 6

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{6-0}{(6+8)-(-2+0)}$$
$$= \frac{6}{16}$$
$$= \frac{3}{8}$$
$$1-p = 1-\frac{3}{8} = \frac{5}{8}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{6-(-2)}{16}$$
$$= \frac{8}{16}$$
$$= \frac{1}{2}$$
$$1-r = 1-\frac{1}{2} = \frac{1}{2}$$

Value of the game

$$V = \frac{ad - bc}{(a+d)-(b+c)}$$
$$= \frac{6x8 - 0x(-2)}{16}$$
$$= \frac{48}{16} = 3$$

So the optimal strategies are

$$A = \left\{\frac{3}{8}, \frac{5}{8}, 0, 0, 0\right\} \text{ and } B = \left\{0, \frac{1}{2}, \frac{1}{2}, 0, 0\right\}.$$

The value of the game = 3. Thus the game is favourable to firm A.

Problem 3

For the game with the following pay-off matrix, determine the saddle point

Player B

$$\text{Player A} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -3 \\ 1 & 0 & 3 & 2 \\ -3 & -2 & -1 & 4 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{ccccc} & \textit{Column II} & & \textit{Column III} & \\ 1 & \boxed{-1} & & \boxed{0} & 0 > -1 \\ 2 & \boxed{0} & & \boxed{3} & 3 > 0 \\ 3 & \boxed{-2} & & \boxed{-1} & -1 > -2 \end{array}$$

The choice is with the player B. He has to choose between strategies II and III. He will lose more in strategy III than in strategy II, irrespective of what strategy is followed by A. So he will drop strategy III and retain strategy II. Now the given game reduces to the following game.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} I & II & IV \end{array} \\ \begin{bmatrix} 2 & -1 & -3 \\ 1 & 0 & 2 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Consider the rows and columns of this matrix.

Row minimum:

I Row	:	-3	
II Row	:	0	Maximum of {-3, 0, -3} = 0
III Row	:	-3	

Column maximum:

I Column	:	2	
II Column	:	0	Minimum of {2, 0, 4} = 0
III Column	:	4	

We see that

Maximum of row minimum = Minimum of column maximum = 0.
So, a saddle point exists for the given game and the value of the game is 0.

Interpretation

No player gains and no player loses. i.e., The game is not favourable to any player. i.e. It is a fair game.

Problem 4

Solve the game

Player B

$$\text{Player A} \quad \begin{bmatrix} 4 & 8 & 6 \\ 6 & 2 & 10 \\ 4 & 5 & 7 \end{bmatrix}$$

Solution

First consider the minimum of each row

Row	**Minimum**
1	4
2	2
3	4

Maximum of $\{4, 2, 4\} = 4$

Next, consider the maximum of each column.

Column	**Maximum**
1	6
2	8
3	10

Minimum of $\{6, 8, 10\} = 6$

Since Maximum of { Row Minima} and Minimum of { Column Maxima } are different, it follows that the given game has no saddle point.

Denote the strategies of player A by A_1, A_2, A_3 . Denote the strategies of player B by B_1, B_2, B_3 . Compare the first and third columns of the given matrix.

$$\begin{array}{cc} B_1 & B_3 \\ \boxed{\begin{matrix} 4 \\ 6 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 10 \\ 7 \end{matrix}} \end{array}$$

The pay-offs in B_3 are greater than or equal to the corresponding pay-offs in B_1. The player B has to make a choice between his strategies 1 and 3. He will lose more if he follows strategy 3 rather than strategy 1. Therefore he will give up strategy 3 and retain strategy 1. Consequently, the given game is transformed into the following game

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 4 & 5 \end{bmatrix} \end{array}$$

Compare the first and third rows of the above matrix

$$\begin{array}{c} \\ A_1 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 4 & 5 \end{bmatrix} \end{array}$$

The pay-offs in A_1 are greater than or equal to the corresponding pay-offs in A_3 . The player A has to make a choice between his strategies 1 and 3. He will gain more if he follows strategy 1 rather than strategy 3. Therefore he will retain strategy 1 and give up strategy 3. Now the given game is transformed into the following game.

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix} \end{array}$$

It is a 2x2 game. Consider the row minima

Row	Minimum
1	4
2	2

Maximum of {4, 2} = 4

Next, consider the maximum of each column

Column	**Maximum**
1	6
2	8

Minimum of {6, 8} = 6

Maximum {row minima} and Minimum {column maxima } are not equal Therefore, the reduced game has no saddle point. So, it is a mixed game

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$. We have a = 4, b = 8, c = 6 and d = 2.

The probability that player A will use his first strategy is p. This is calculated as

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{2-6}{(4+2)-(8+6)}$$
$$= \frac{-4}{6-14}$$
$$= \frac{-4}{-8} = \frac{1}{2}$$

The probability that player B will use his first strategy is r. This is calculated as

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{2-8}{-8}$$
$$= \frac{-6}{-8}$$
$$= \frac{3}{4}$$

Value of the game is V. This is calculated as

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{4x2-8x6}{-8}$$
$$= \frac{8-48}{-8}$$
$$= \frac{-40}{-8} = 5$$

Interpretation

Out of 3 trials, player A will use strategy 1 once and strategy 2 once. Out of 4 trials, player B will use strategy 1 thrice and strategy 2 once. The game is favourable to player A.

Problem 5

Dividing a game into sub-games
Solve the game with the following pay-off matrix.

Player B

$$\text{Player A} \quad \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -4 & 6 & 3 \\ -3 & 3 & 4 \\ 2 & -3 & 4 \end{bmatrix} \end{array}$$

Solution

First, consider the row mimima

Row	Minimum
1	-4
2	-3
3	-3

Maximum of {-4, -3, -3} = -3

Next, consider the column maxima.

Column	Maximum
1	2
2	6
3	4

Minimum of {2, 6, 4} = 2

We see that Maximum of { row minima} $\neq$ Minimum of { column maxima}.

So the game has no saddle point. Hence it is a mixed game. Compare the first and third columns

$$\begin{array}{ccc} \textit{I Column} & \textit{III Column} & \\ \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix} & \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} & \begin{array}{r} -4 \leq 3 \\ -3 \leq 4 \\ 2 \leq 4 \end{array} \end{array}$$

We assert that Player B will retain the first strategy and give up the third strategy. We get the following reduced matrix.

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \\ 2 & -3 \end{bmatrix}$$

We check that it is a game with no saddle point.

Sub games

Let us consider the 2x2 sub games. They are:

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$$

First, take the sub game

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$$

Compare the first and second columns. We see that $-4 \leq 6$ and $-3 \leq 3$. Therefore, the game reduces to $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Since $-4 < -3$, it further reduces to −3.

Next, consider the sub game

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

We see that it is a game with no saddle point. Take a = -4, b = 6, c = 2, d = -3. Then the value of the game is

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

Next, take the sub game $\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$. In this case we have a = -3, b = 3, c = 2 and d = -3. The value of the game is obtained as

$$V = \frac{ad - bc}{(a+d)-(b+c)}$$

$$= \frac{(-3)(-3)-(3)(2)}{(-3-3)-(3+2)}$$

$$= \frac{9-6}{-6-5} = -\frac{3}{11}$$

Let us tabulate the results as follows:

Sub game	**Value**
$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$	-3
$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$	0
$\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$	$-\frac{3}{11}$

The value of 0 will be preferred by the player A. For this value, the first and third strategies of A correspond while the first and second strategies of the player B correspond to the value 0 of the game. So it is a fair game.

Questions

1. Explain the principle of dominance in the theory of games.
2. Explain how a game can be solved through sub games.
3. Solve the following game by the principle of dominance:

Player B

		I	*II*	*III*	*IV*
Player A Strategies	1	8	10	9	14
	2	10	11	8	12
	3	13	12	14	13

Answer: V = 12

4. Solve the game by the principle of dominance:

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

Answer: V = 4

5. Solve the game with the following pay-off matrix

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

Answer : $p = \frac{3}{5}, r = \frac{2}{5}, V = -\frac{11}{5}$

6. Solve the game

$$\begin{bmatrix} 8 & 7 & 6 & -1 & 2 \\ 12 & 10 & 12 & 0 & 4 \\ 14 & 6 & 8 & 14 & 16 \end{bmatrix}$$

Answer : $p = \frac{4}{9}, r = \frac{7}{9}, V = \frac{70}{9}$

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Lesson 5 - Graphical Solution Of A 2X2 Game With No Saddle Point

Lesson Outline

- The principle of graphical solution
- Numerical example

Learning Objectives

After reading this lesson you should be able to

- Understand the principle of graphical solution
- Derive the equations involving probability and expected value
- Solve numerical problems

Example:

Consider the game with the following pay-off matrix.

Player B

$$\text{Player A} \quad \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

First consider the row minima.

Row	Minimum
1	2
2	1

Maximum of {2, 1} = 2.

Next, consider the column maxima.

Column	Maximum
1	4
2	5

Minimum of {4, 5} = 4.

We see that Maximum { row minima} $\neq$ Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Equations involving probability and expected value

Let p be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is 1-p.

Let E be the expected value of pay-off to player A.

When B uses his first strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 2p + 4(1-p) \\ &= 2p + 4 - 4p \\ &= 4 - 2p \end{aligned} \qquad (1)$$

When B uses his second strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 5p + 1(1-p) \\ &= 5p + 1 - p \\ &= 4p + 1 \end{aligned} \qquad (2)$$

Consider equations (1) and (2). For plotting the two equations on a graph sheet, get some points on them as follows:

E = -2p+4

p	0	1	0.5
E	4	2	3

E = 4p+1

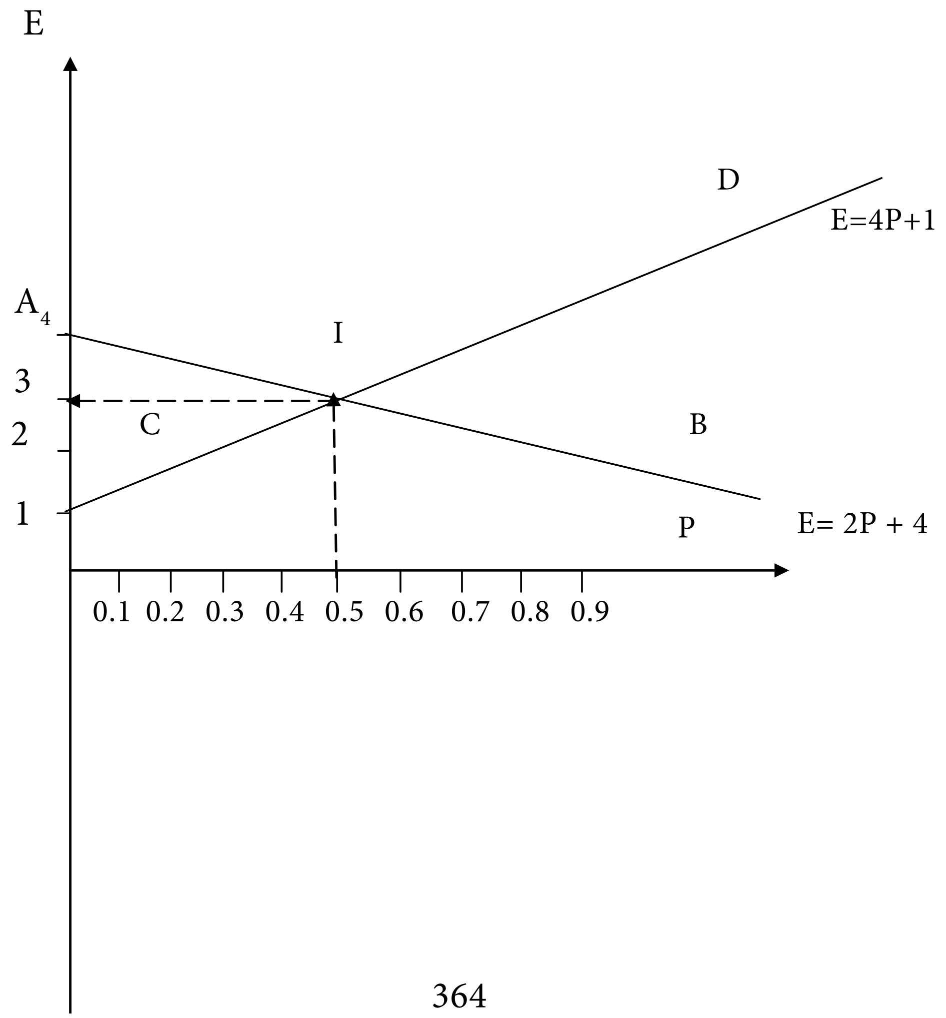
p	0	1	0.5
E	1	5	3

Graphical solution

Procedure

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will give the common solution of the two equations (1) and (2). Thus we would obtain the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = 0.5 and E = 3. Therefore, the value V of the game is 3.



Problem 1

Solve the following game by graphical method.

Player B

$$\text{Player A} \quad \begin{bmatrix} -18 & 2 \\ 6 & -4 \end{bmatrix}$$

Solution

First consider the row minima

Row	Minimum
1	-18
2	-4

Maximum of {-18, - 4} = - 4.

Next, consider the column maxima

Column	Maximum
1	6
2	2

Minimum of {6, 2} = 2

We see that Maximum { row minima} $\neq$ Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Let p be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is 1-p.

When B uses his first strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= -18p + 6(1-p) \\ &= -18p + 6 - 6p \\ &= -24p + 6 \end{aligned} \qquad (1)$$

When B uses his second strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 2p - 4(1-p) \\ &= 2p - 4 + 4p \\ &= 6p - 4 \end{aligned} \qquad (2)$$

Consider equations (1) and (2). For plotting the two equations on a graph sheet, get some points on them as follows:

E = -24 p + 6

p	0	1	0.5
E	6	-18	-6

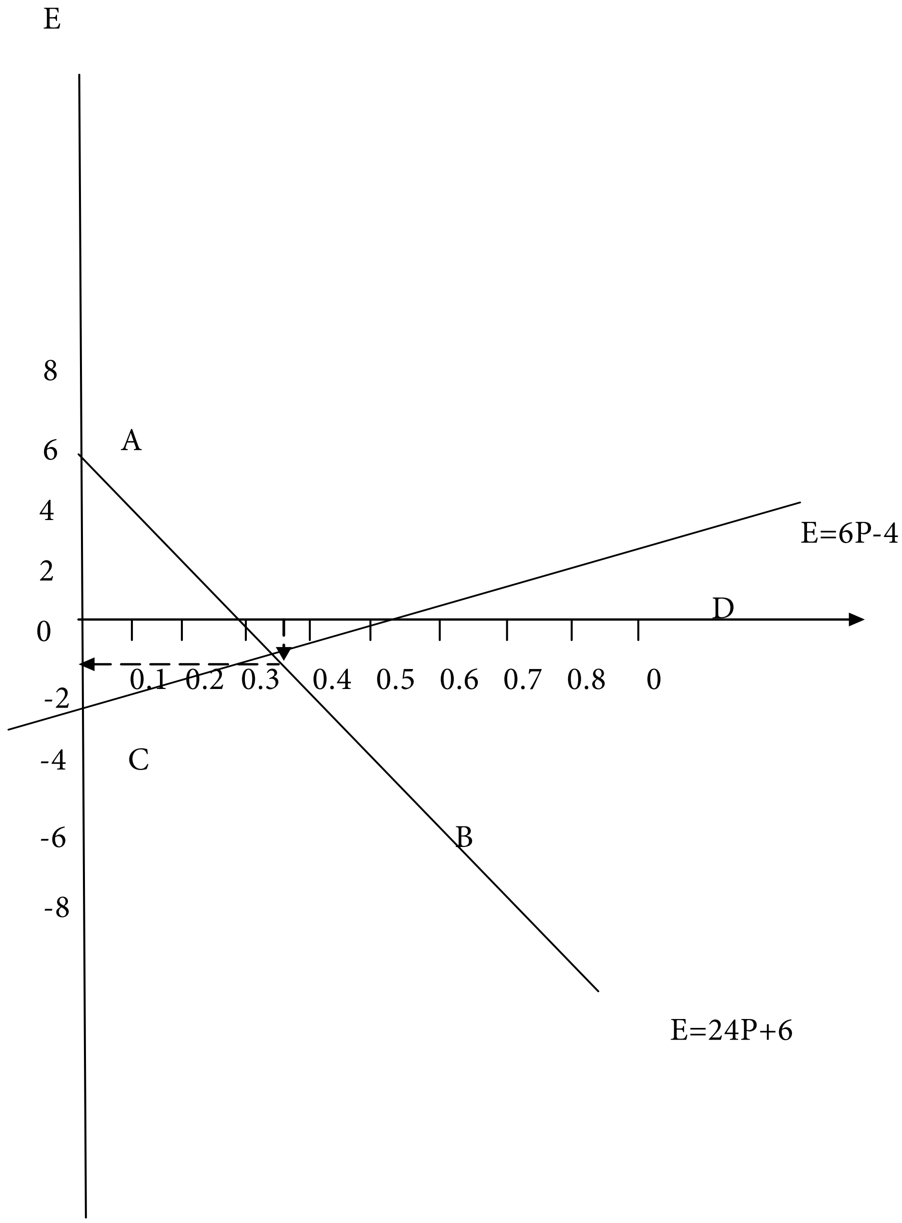
E = 6p-4

p	0	1	0.5
E	-4	2	-1

Graphical solution

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will provide the common solution of the two equations (1) and (2). Thus we would get the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = $\frac{1}{3}$ and E = -2. Therefore, the value V of the game is -2.



Questions

1. Explain the method of graphical solution of a 2x2 game.

2. Obtain the graphical solution of the game

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer: $p = \frac{1}{2}$, $V = 9$

3. Graphically solve the game

$$\begin{bmatrix} 4 & 10 \\ 8 & 6 \end{bmatrix}$$

Answer: $p = \frac{1}{4}$, $V = 7$

4. Find the graphical solution of the game

$$\begin{bmatrix} -12 & 12 \\ 2 & -6 \end{bmatrix}$$

Answer: $p = \frac{1}{4}$, $V = -\frac{3}{2}$

5. Obtain the graphical solution of the game

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer: $p = \frac{1}{2}$, $V = 9$

6. Graphically solve the game

$$\begin{bmatrix} -3 & -5 \\ -5 & 1 \end{bmatrix}$$

Answer: $p = \frac{3}{4}$, $V = -\frac{7}{2}$

Lesson 6 - 2 X N Zero-Sum Games

Lesson Outline

- A 2 X N Zero-Sum Game
- Method Of Solution
- Sub Game Approach And Graphical Method
- Numerical Example

Learning Objectives

After reading this lesson you should be able to

- Understand the concept of a 2 x n zero-sum game
- Solve numerical problems

The concept of a 2 x n zero-sum game

When the first player A has exactly two strategies and the second player B has n (where n is three or more) strategies, there results a 2 x n game. It is also called a rectangular game. Since A has two strategies only, he cannot try to give up any one of them. However, since B has many strategies, he can make out some choice among them. He can retain some of the advantageous strategies and discard some disadvantageous strategies. The intention of B is to give as minimum payoff to A as possible. In other words, B will always try to minimize the loss to himself. Therefore, if some strategies are available to B by which he can minimize the payoff to A, then B will retain such strategies and give such strategies by which the payoff will be very high to A.

Approaches for 2 x n zero-sum game

There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.

Sub game approach

The given 2 x n game is divided into 2 x 2 sub games. For this purpose, consider all possible 2 x 2 sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since B can make out a choice of his strategies, he will discard such of those sub games which result in more payoff to A. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

Problem

Solve the following game

Player B

$$\text{Player A} \quad \begin{bmatrix} 8 & -2 & -6 & 9 \\ 3 & 5 & 10 & 2 \end{bmatrix}$$

Solution

Let us consider all possible 2x2 sub games of the given game. We have the following sub games:

1.

$$\begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix}$$

2.

$$\begin{bmatrix} 8 & -6 \\ 3 & 10 \end{bmatrix}$$

3.

$$\begin{bmatrix} 8 & 9 \\ 3 & 2 \end{bmatrix}$$

4.

$$\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$$

5.

$$\begin{bmatrix} -2 & 9 \\ 5 & 2 \end{bmatrix}$$

6.

$$\begin{bmatrix} -6 & 9 \\ 10 & 2 \end{bmatrix}$$

Let E be the expected value of the pay off to player A. Let p be the probability that player A will use his first strategy. Then the probability that he will use his second strategy is 1-p. We form the equations for E in all the sub games as follows:

Sub game (1)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$
Equation 2: $E = -2p + 5(1-p) = -7p + 5$

Sub game (2)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$
Equation 2: $E = -6p + 10(1-p) = -16p + 10$

Sub game (3)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$
Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Sub game (4)

Equation 1: $E = -2p + 5(1-p) = -7p + 5$
Equation 2: $E = -6p + 10(1-p) = -16p + 10$

Sub game (5)

Equation 1: $E = -2p + 5(1-p) = -7p + 5$
Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Sub game (6)

Equation 1: $E = -6p + 10(1-p) = -16p + 10$
Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Solve the equations for each sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game	p	Expected value E
1	1/6	23/6
2	1/3	14/3
3	1/2	11/2
4	5/9	10/9
5	3/14	7/2
6	8/23	102/23

Interpretation

Since player A has only 2 strategies, he cannot make any choice on the strategies. On the other hand, player B has 4 strategies. Therefore he can retain any 2 strategies and give up the other 2 strategies. This he will do in such a way that the pay-off to player A is at the minimum. The pay-off to A is the minimum in the case of sub game 4. i.e., the sub game with the matrix .

$$\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$$

Therefore, in the long run, player B will retain his strategies 2 and 3 and give up his strategies 1 and 4. In that case, the probability that A will use his first strategy is p = 5/9 and the probability that he will use his second strategy is 1-p = 4/9 . i.e., Out of a total of 9 trials, he will use his first strategy five times and the second strategy four times. The value of the game is10/9 . The positive sign of V shows that the game is favourable to player A.

Graphical Solution

Now we consider the graphical method of solution to the given game.

Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

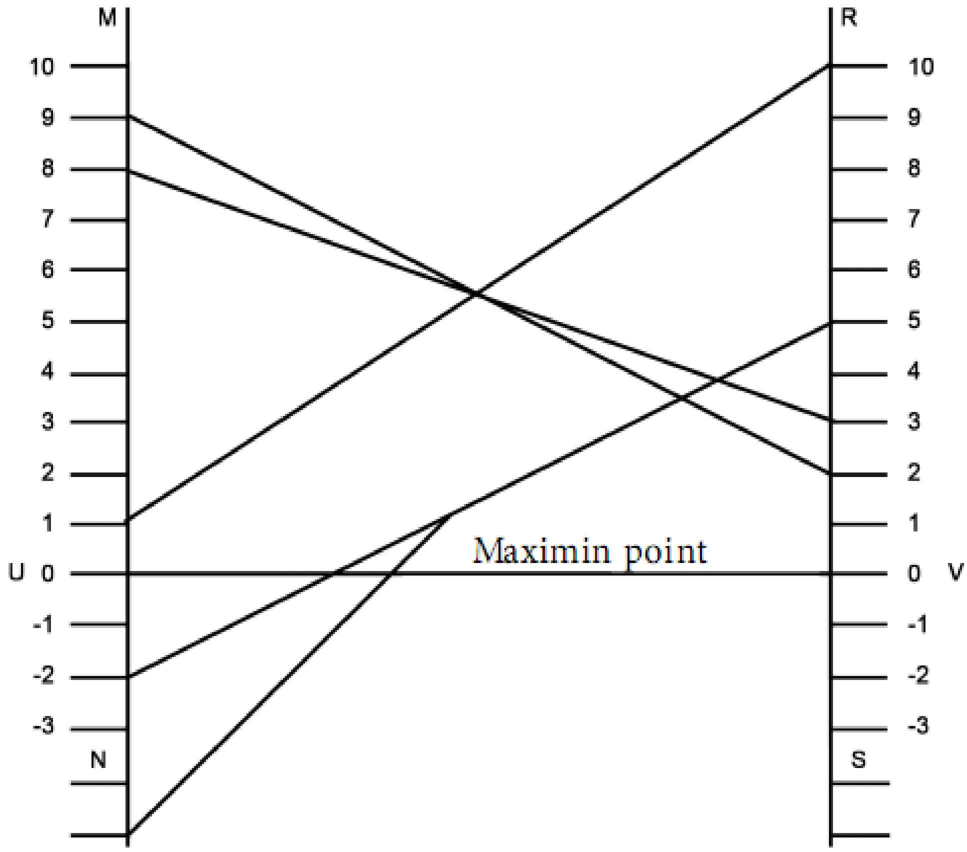
We have to plot the following points: (8, 3), (-2, 5), (-6, 10), (9, 2). The points 8, -2, -6, 9 are marked on MN. The points 3, 5, 10, 2 are marked on RS.

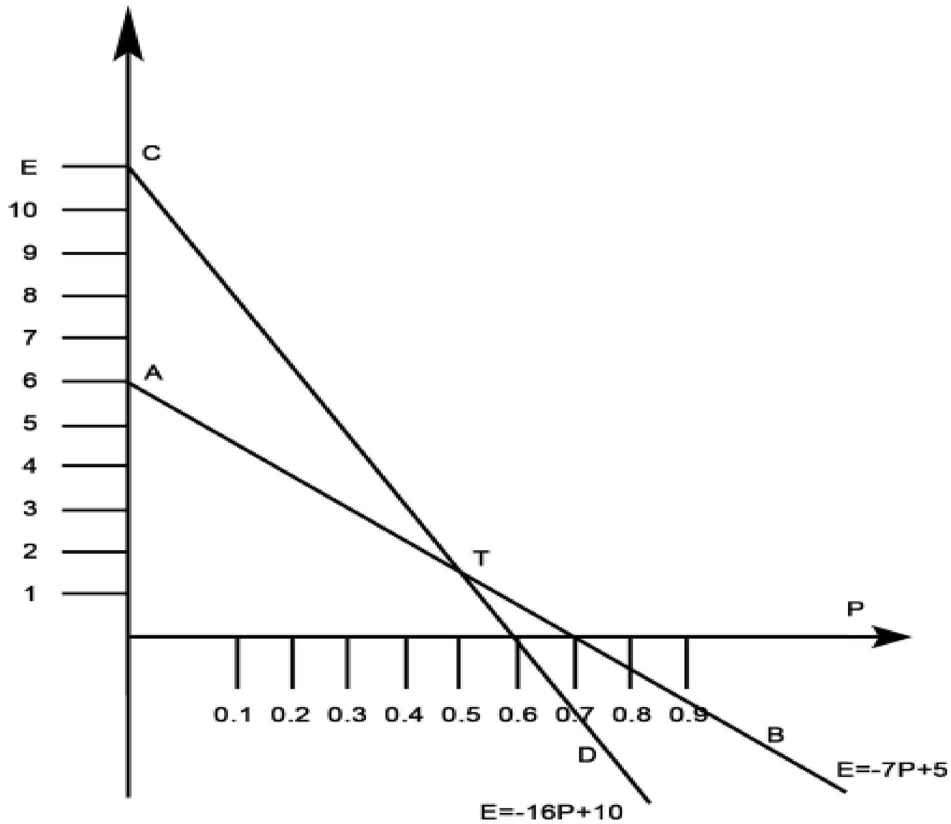
Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 8 on MN with the point 3 on RS. We have 4 such straight lines. They represent the 4 moves of the second player. They intersect in 6 points. Take the lowermost point of intersection of the straight lines. It is called the Maximin point. With the help of this point, identify the optimal strategies for the second player. This point corresponds to the points –2 and –6 on MN and 5 and 10 on RS. They correspond to the sub game with the matrix .

$$\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$$

The points –2 and –6 on MN correspond to the second and third strategies of the second player. Therefore, the graphical method implies that, in the long run, the second player will retain his strategies 2 and 3 and give up his strategies 1 and 4.

We graphically solve the sub game with the above matrix. We have to solve the two equations E = -7 p + 5 and E = - 16 p + 10. Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p =5/9 and E = 10/9 . Therefore, the value V of the game is 10/9 . We see that the probability that first player will use his first strategy is p = 8/9 and the probability that he will use his second strategy is 1-p = 4/9 .





$E = -7p+5$

p	0	1	0.5
E	5	- 2	1.5

$E = -16p+10$

p	0	1	0.5
E	10	- 6	2

Questions

1. Explain a 2 x n zero-sum game.
2. Describe the method of solution of a 2 x n zero-sum game.
3. Solve the following game:

Player B

$$\text{Player A} \quad \begin{bmatrix} 10 & 2 & 6 \\ 1 & 5 & 8 \end{bmatrix}$$

Answer: p = , V = 4t

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Lesson 7 - M X 2 Zero-Sum Games

Lesson Outline

- An m x 2 zero-sum game
- Method of solution
- Sub game approach and graphical method
- Numerical example

Learning Objectives

After reading this lesson you should be able to

- Understand the concept of an m x 2 zero-sum game
- Solve numerical problems

The concept of an m x 2 zero-sum game

When the second player B has exactly two strategies and the first player A has m (where m is three or more) strategies, there results an m x 2 game. It is also called a rectangular game. Since B has two strategies only, he will find it difficult to discard any one of them. However, since A has more strategies, he will be in a position to make out some choice among them. He can retain some of the most advantageous strategies and give up some other strategies. The motive of A is to get as maximum payoff as possible. Therefore, if some strategies are available to A by which he can get more payoff to himself, then he will retain such strategies and discard some other strategies which result in relatively less payoff.

Approaches for m x 2 zero-sum game

There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.

Sub game approach

The given m x 2 game is divided into 2 x 2 sub games. For this purpose, consider all possible 2 x 2 sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since A can make out a choice of his strategies, he will be interested in such of those sub games which result in more payoff to himself. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

Problem

Solve the following game

Player B
Strategies

Player A Strategies		*I*	*II*
	1	5	8
	2	−2	10
	3	12	4
	4	6	5

Solution

Let us consider all possible 2x2 sub games of the given game. We have the following sub games:

7. $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$ 8. $\begin{bmatrix} 5 & 8 \\ 12 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 5 & 8 \\ 6 & 5 \end{bmatrix}$ 10. $\begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix}$ 12. $\begin{bmatrix} 12 & 4 \\ 6 & 5 \end{bmatrix}$

Let E be the expected value of the payoff to player A. i.e., the loss to player B. Let r be the probability that player B will use his first strategy. Then the probability that he will use his second strategy is 1-r. We form the equations for E in all the sub games as follows:

Sub game (1)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = -2r + 10(1-r) = -12r + 10$

Sub game (2)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = 12r + 4(1-r) = 8r + 4$

Sub game (3)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Sub game (4)

Equation 1: $E = -2r + 10(1-r) = -12r + 10$

Equation 2: $E = 12r + 4(1-r) = 8r + 4$

Sub game (5)

Equation 1: $E = -2r + 10(1-r) = -12r + 10$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Sub game (6)

Equation 1: $E = 12r + 4(1-r) = 8r + 4$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Solve the equations for each 2x2 sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game

Sub game	R	Expected value E
1	2/9	22/9
2	4/11	76/11
3	3/4	23/4
4	3/10	32/5
5	5/13	70/13
6	1/7	36/7

Interpretation

Since player B has only 2 strategies, he cannot make any choice on his strategies. On the other hand, player A has 4 strategies and so he can retain any 2 strategies and give up the other 2 strategies. Since the choice is with A, he will try to maximize the payoff to himself. The pay-off to A is the maximum in the case of sub game 1. i.e., the sub game with the matrix .

$$\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$$

Therefore, player A will retain his strategies 1 and 2 and discard his strategies 3 and 4, in the long run. In that case, the probability that B will use his first strategy is r = 2/9 and the probability that he will use his second strategy is 1-r = 7/9 . i.e., Out of a total of 9 trials, he will use his first strategy two times and the second strategy seven times.

The value of the game is 22/3 . The positive sign of V shows that the game is favourable to player A.

Graphical Solution

Now we consider the graphical method of solution to the given game.

Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

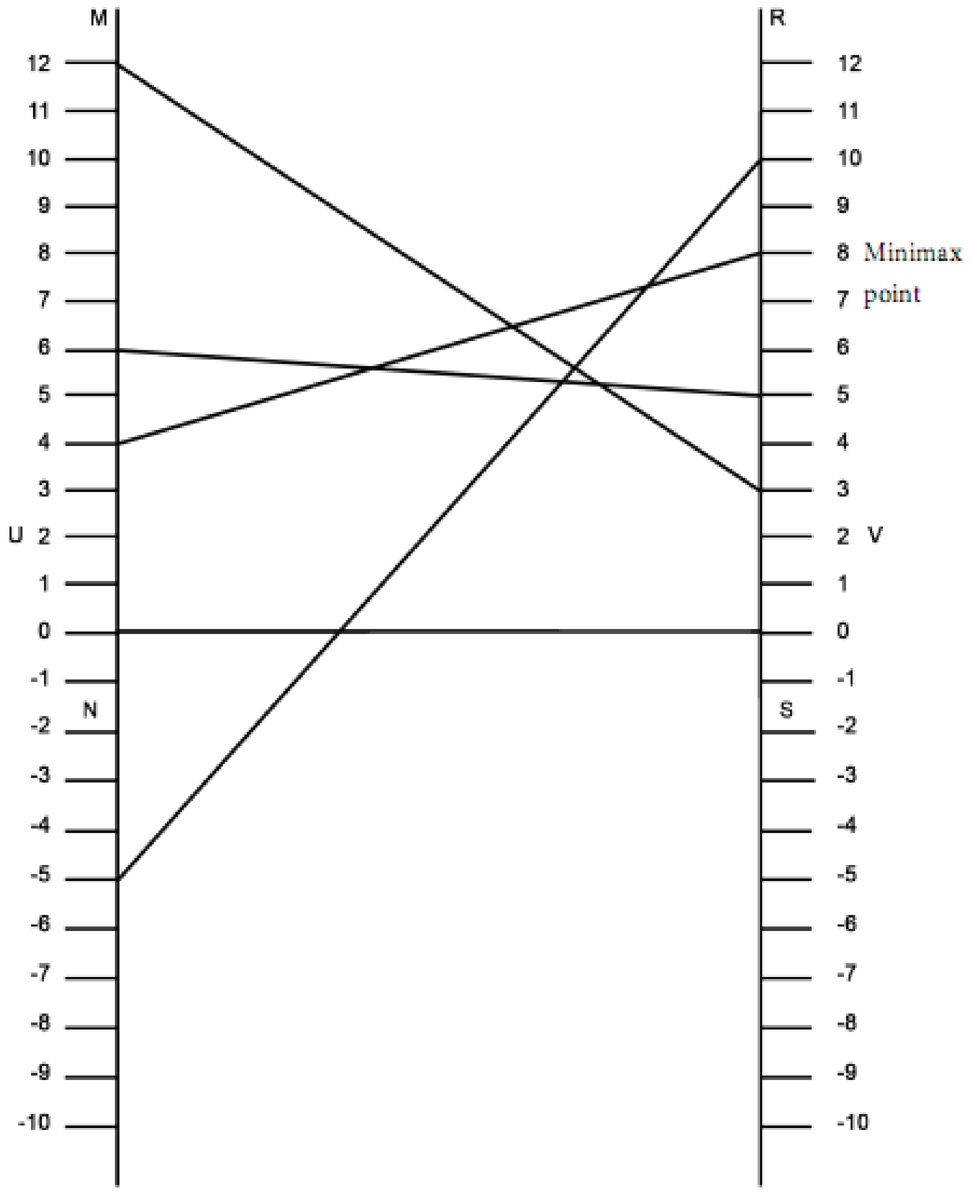
We have to plot the following points: (5, 8), (-2, 10), (12, 4), (6, 5).The points 5, -2, 12, 6 are marked on MN. The points 8, 10, 4, 5 are marked on RS.

Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 5 on MN with the point 8 on RS. We have 4 such straight lines. They represent the 4 moves of the first player. They intersect in 6 points. Take the uppermost point of intersection of the straight lines. It is called the Minimax point. With the help of this point, identify the optimal strategies for the first player. This point corresponds to the points 5 and -2 on MN and 8 and 10 on RS. They correspond to the sub game with the matrix

$$\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$$

. The points 5 and -2 on MN correspond to the first and second strategies of the first player. Therefore, the graphical method implies that the first player will retain his strategies 1 and 2 and give up his strategies 3 and 4, in the long run.

We graphically solve the sub game with the above matrix. We have to solve the two equations E = - 3 r + 8 and E = - 12 r + 10. Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have r = 2/9 and E = 22/3. Therefore, the value V of the game is 22/3 . We see that the probability that the second player will use his first strategy is r = 2/9 and the probability that he will use his second strategy is 1-r = 7/9 .


M
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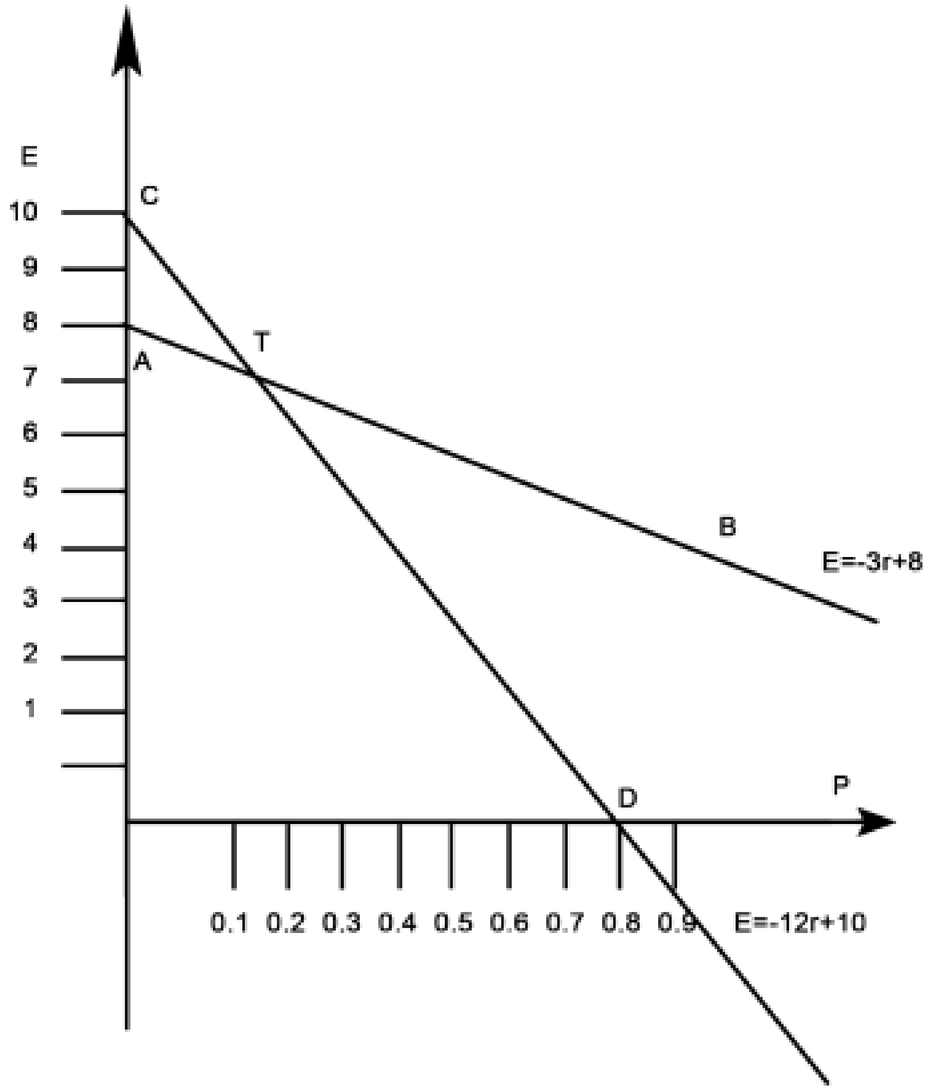

E = - 3r+8

p	0	1	0.5
E	8	5	6.5

E = - 12r+10

p	0	1	0.5
E	10	- 2	4

Questions

1. What is an m x 2 zero-sum game? Explain.
2. How will you solve an m x 2 zero-sum game? Explain.
3. Solve the following game:

Player B

Strategies

		I	II
Player A Strategies	1	20	8
	2	5	2
	3	8	12

Answer: r = , V = 11

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Lesson 8 - Linear Programming Approach To Game Theory

Lesson Outline

- How to solve a game with LPP?
- Formulation of LPP
- Solution by simplex method

Learning Objectives

After reading this lesson you should be able to

- Understand the transformation of a game into LPP
- Solve a game by simplex method

Introduction

When there is neither saddle point nor dominance in a problem of game theory and the payoff matrix is of order 3x3 or higher, the probability and graphical methods cannot be employed. In such a case, linear programming approach may be followed to solve the game.

Linear programming technique

A general approach to solve a game by linear programming technique is presented below. Consider the following game:

Player B

$$\text{Player A} \quad \begin{array}{c} \\ X_1 \\ X_2 \\ \\ X_m \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \end{array}$$

It is required to determine the optimal strategy for A = $\{X_1, X_2, ... X_m\}$ and B = $\{Y_1, Y_2, ... Y_n\}$. First we shall determine the optimal strategies of player B.

If player A adopts strategy , then the expected value of loss to B is
,

$$a_{11}Y_1 + a_{12}Y_2 + ... + a_{1n}Y_n \leq V$$

where V is the value of game. If A adopts strategy X_2 , then the expected value of loss to B is

$$a_{21}Y_1 + a_{22}Y_2 + ... + a_{2n}Y_n \leq V$$

and so on. Also we have

$$Y_1 + Y_2 + ... + Y_n = 1$$

and

$Y_j \geq 0$ for all j.
Without loss of generality, we can assume that V > 0. Divide each of the above relation by V and let $Y'_j = \frac{Y_j}{V}$.

Then we have

$$\sum Y'_j = \sum \frac{Y_j}{V} = \frac{1}{V}$$

From this we obtain

$$a_{11}Y'_1 + a_{12}Y'_2 + ... + a_{1n}Y'_n \leq 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + ... + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + ... + a_{mn}Y'_n \leq 1$$

and

$$Y'_1 + Y'_2 + \ldots + Y'_n = \frac{1}{V}$$

with $Y'_j \geq 0$ for all j.

The objective of player B is to minimise the loss to himself . Thus the problem is to minimize V, or equivalently to maximise $\frac{1}{V}$. Therefore, the objective of player B is to maximise the value of $Y'_1 + Y'_2 + \ldots + Y'_n$ subject to the m linear constraints provided above.

Statement of the problem

Maximise: $Y'_1 + Y'_2 + \ldots + Y'_n$, subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + \ldots + a_{1n}Y'_n \leq 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + \ldots + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + \ldots + a_{mn}Y'_n \leq 1$$

$$Y'_1, Y'_2, \ldots, Y'_n \geq 0.$$

We can use simplex method to solve the above problem. For this purpose, we have to introduce non-negative slack variables $s_1, s_2, \ldots, s_m$ to each of the inequalities. So the problem can be restated as follows:
Restatement of the problem:

Maximise: $Y'_1 + Y'_2 + \ldots + Y'_n + 0s_1 + 0s_2 + \ldots + 0s_m$ subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + \ldots + a_{1n}Y'_n + s_1 = 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + \ldots + a_{2n}Y'_n + s_2 = 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + \ldots + a_{mn}Y'_n + s_m = 1$$

with $Y_j = Y'_j V$ for all j and $s_1 \geq 0,\ s_2 \geq 0, \ldots, s_m \geq 0$.
Thus we get the optimal strategy for player B to be ($Y_1, Y_2, \ldots, Y_n$).
In a similar manner we can determine the optimal strategy for player A.

Application

We illustrate the method for a 2X2 zero sum game.

Problem 1
Solve the following game by simplex method for LPP:

Player B

Player A $\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix}$

Solution

Row minima : I row : 3
II row : 2
Maximum of {3,2} = 3

Column maxima: I column : 5
II column : 6
Minimum of {5,6} = 5

So, Maximum of {Row minima} $\neq$ Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

Problem formulation

Let V denote the value of the game. Let the probability that the player B will use his first strategy be r and second strategy be s. Let V denote the value of the game.

When A follows his first strategy

The expected payoff to A (i.e., the expected loss to B) = 3 r + 6 s.

This pay-off cannot exceed V. So we have =3 r + 6 s V (1)

When A follows his second strategy

The expected pay-off to A (i.e., expected loss to B) = 5 r + 2 s.

This cannot exceed V. Hence we obtain the condition = 5 r + 2 s V (2)

From (1) and (2) we have

$$3\frac{r}{V}+6\frac{s}{V}\le 1$$
$$and\ \ 5\frac{r}{V}+2\frac{s}{V}\le 1$$

Substitute $\frac{r}{V}=x, \frac{s}{V}=y.$

Then we have

$$3x+6y\le 1$$
$$and\ \ 5x+2y\le 1$$

where r and s are connected by the relation

$$r+s=1$$

.

$$\text{i.e., } \frac{r}{V}+\frac{s}{V}=\frac{1}{V}$$
$$\text{i.e., } x+y=\frac{1}{V}$$

B will try to minimise V. i.e., He will try to maximise $\frac{1}{V}$. Thus we have the following LPP.

Maximize $\frac{1}{V}=x+y$,

subject to the restrictions

$$3x+6y\le 1,$$
$$5x+2y\le 1,$$
$$x\ge 0,\ \ y\ge 0$$

Solution of LPP

Introduce two slack variables s_1, s_2 . Then the problem is transformed into the
following one:

Maximize $\frac{1}{V} = x + y + 0.s_1 + 0.s_2$

subject to the constraints

$$3x + 6y + 1.s_1 + 0.s_2 = 1,$$
$$5x + 2y + 0.s_1 + 1.s_2 = 1,$$
$$x \geq 0,\ y \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$$

Let us note that the above equations can be written in the form of a single matrix equation as

A X = B

where A = $\begin{bmatrix} 3 & 6 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix}$, B = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The entries in B are referred to as the b – values. Initially, the basic variables are s_1, s_2 . We have the following simplex tableau:

	x	y	s_1	s_2	b – value
s_1 - row	3	6	1	0	1
s_2 - row	5	2	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are –1, -1. The absolute values are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable x. We take the new basic variable as x. Consider the ratio of b-value to x-value. We have the following ratios:

s_1 - row : $\frac{1}{3}$

s_2 - row : $\frac{1}{5}$

Minimum of { $\frac{1}{3}$, $\frac{1}{5}$} = $\frac{1}{5}$.

Hence select as the leaving variable. Thus the pivotal element is 5. We obtain the following tableau at the end of Iteration No. 1.

	x	y	s_1	s_2	b – value
s_1 - row	0	$\frac{24}{5}$	1	$-\frac{3}{5}$	$\frac{2}{5}$
x - row	1	$\frac{2}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$
Objective function row	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$

Now, the negative element in the objective function row is $-\frac{3}{5}$. This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

s_1 - row : $\left(\frac{2}{5}\right) \Big/ \left(\frac{24}{5}\right) = \frac{1}{12}$

x - row : $\left(\frac{1}{5}\right) \Big/ \left(\frac{2}{5}\right) = \frac{1}{2}$

Minimum of $\left\{\frac{1}{12}, \frac{1}{2}\right\} = \frac{1}{12}$

Hence select s_1 as the leaving variable. The pivotal element is $\frac{24}{5}$. We get the following tableau at the end of Iteration No. 2.

	x	y	s_1	s_2	b – value
y - row	0	1	$\frac{5}{24}$	$-\frac{1}{8}$	$\frac{1}{12}$
x - row	1	0	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$
Objective function row	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of $\frac{1}{V}$ is $\frac{1}{4}$. This is provided by x = $\frac{1}{6}$ and y = $\frac{1}{12}$. Thus the optimum value of the game is obtained as V = 4. Using the relations $\frac{r}{V}=x, \frac{s}{V}=y$, we obtain $r=\frac{4}{6}=\frac{2}{3}$ and $s=\frac{4}{12}=\frac{1}{3}$.

Problem 2

Solve the following game

Player B

Player A $\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$

Solution

The game has no saddle point. It is a mixed game. Let the probability that B will use his first strategy be r. Let the probability that B will use his second strategy be s. Let V be the value of the game.

When A follows his first strategy

The expected payoff to A (i.e., the expected loss to B) = 2 r +5 s.
The pay-off to A cannot exceed V. So we have = 2 r + 5 s V (I)

When A follows his second strategy

The expected pay-off to A (i.e., expected loss to B) = 4 r + s. The pay-off to A cannot exceed V. Hence we obtain the condition

= 4 r + s V (II)

From (I) and (II) we have

$$2\frac{r}{V}+5\frac{s}{V}\le 1$$

$$and \;\; 4\frac{r}{V}+\frac{s}{V}\le 1$$

Substitute

$$\frac{r}{V}=x \;\; and \;\; \frac{s}{V}=y.$$

Thus we have

$$2x+5y\le 1$$

$$and \;\; 4x+y\le 1$$

where r and s are connected by the relation

$$r+s=1$$

.

i.e., $$\frac{r}{V}+\frac{s}{V}=\frac{1}{V}$$

i.e., $$x+y=\frac{1}{V}$$

The objective of B is to minimise V. i.e., He will try to maximise $\frac{1}{V}$. Thus we are led to the following linear programming problem:

Maximize $\frac{1}{V}$ = x + y

subject to the constraints

$$2x+5y\le 1,$$
$$4x+y\le 1,$$
$$x\ge 0,\; y\ge 0.$$

To solve this linear programming problem, we use simplex method as detailed below.

Introduce two slack variables s_1, s_2 . Then the problem is transformed into the following one:

Maximize $$\frac{1}{V}=x+y+0.s_1+0.s_2$$

subject to the constraints

$$2x + 5y + 1.s_1 + 0.s_2 = 1,$$
$$4x + y + 0.s_1 + 1.s_2 = 1,$$
$$x \geq 0,\ y \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$$

We have the following simplex tableau:

	x	y	s_1	s_2	b – value
s_1 - row	2	5	1	0	1
s_2 - row	4	1	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are –1, -1. The absolute value are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable x. We take the new basic variable as x. Consider the ratio of b-value to x-value. We have the following ratios:

s_1 - row : $\frac{1}{2}$

s_2 - row : $\frac{1}{4}$

Minimum of $\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}$

Hence select s_2 as the leaving variable. Thus the pivotal element is 4. We obtain the following tableau at the end of Iteration No. 1.

	x	y	s_1	s_2	b – value
s_1 - row	0	$\frac{9}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
x - row	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Objective function row	0	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$

Now, the negative element in the objective function row is $-\frac{3}{4}$. This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

s_1 - row : $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{9}{2}\right)} = \dfrac{1}{9}$

x - row : $\dfrac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)} = 1$

Minimum of $\left\{\frac{1}{9}, 1\right\} = \frac{1}{9}$

Hence select s_1 as the leaving variable. The pivotal element is $\frac{9}{2}$. We get the following tableau at the end of Iteration No. 2.

	x	y	s_1	s_2	b – value
y - row	0	1	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{1}{9}$
x - row	1	0	$-\frac{1}{18}$	$\frac{5}{18}$	$\frac{2}{9}$
Objective function row	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of $\frac{1}{V}$ is $\frac{1}{3}$.

This is provided by x = $\frac{2}{9}$ and y = $\frac{1}{9}$. Thus the optimum value of the game is got as V = 3. Using the relations $\frac{r}{V} = x$, $\frac{s}{V} = y$, we obtain $r = \frac{6}{9} = \frac{2}{3}$ and $s = \frac{3}{9} = \frac{1}{3}$.

Problem 3

Solve the following game by simplex method for LPP

Player B

Player A $\begin{bmatrix} -48 & 2 \\ 6 & -4 \end{bmatrix}$

Solution

Row minima : I row : -48

II row : -4

Maximum of {-48, -4} = -4

Column maxima: I column : 6

II column : 2

Minimum of (6, 2} = 2

So, Maximum of {Row minima} ≠ Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

Problem formulation

Let V denote the value of the game. Let the probability that the player B will use his first strategy be r and second strategy be s. Let V denote the value of the game.

When A follows his first strategy

The expected payoff to A (i.e., the expected loss to B) = - 48 r + 2 s.

This pay-off cannot exceed V. So we have =- 48 r + 2 s V (1)'

When A follows his second strategy

The expected pay-off to A (i.e., expected loss to B) = 6 r - 4 s.

This cannot exceed V. Hence we obtain the condition = 6 r - 4 s V (2)'

From (1)' and (2)' we have

$$-48\frac{r}{V}+2\frac{s}{V}\leq 1$$

and

$$6\frac{r}{V}-4\frac{s}{V}\leq 1$$

Substitute $\frac{r}{V} = x,\ \frac{s}{V} = y$.

Thus we have
$$-48x + 2y \le 1$$
$$\textit{and}\ \ 6x - 4y \le 1$$

where r and s are connected by the relation
$$r + s = 1 \ .$$

i.e., $\frac{r}{V} + \frac{s}{V} = \frac{1}{V}$

i.e., $x + y = \frac{1}{V}$

B will try to minimise V. i.e., He will try to maximise $\frac{1}{V}$. Thus we have the following LPP.

Maximize $\frac{1}{V} = x + y$,

subject to the restrictions
$$-48x + 2y \le 1,$$
$$6x - 4y \le 1,$$
$$x \ge 0,\ \ y \ge 0$$

Solution of LPP

Introduce two slack variables s_1, s_2 . Then the problem is transformed into the following one:

Maximize $\frac{1}{V} = x + y + 0.s_1 + 0.s_2$

subject to the constraints
$$-48x + 2y + 1.s_1 + 0.s_2 = 1,$$
$$6x - 4y + 0.s_1 + 1.s_2 = 1,$$
$$x \ge 0,\ y \ge 0,\ s_1 \ge 0,\ s_2 \ge 0$$

Initially, the basic variables are s_1, s_2 . We have the following simplex tableau:

	x	y	s_1	s_2	b – value
s_1 - row	-48	2	1	0	1
s_2 - row	6	-4	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are –1, -1. The absolute value are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable x. We take the new basic variable as x. Consider the ratio of b-value to x-value. We have the following ratios:

s_1 - row : $-\frac{1}{48}$

s_2 - row: $\frac{1}{6}$

Minimum of $\{-\frac{1}{48}, \frac{1}{6}\} = -\frac{1}{48}$.

Hence select s_1 as the leaving variable. Thus the pivotal element is - 48. We obtain the following tableau at the end of Iteration No. 1.

	x	y	s_1	s_2	b – value
x - row	1	$-\frac{1}{24}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$
s_2 - row	0	$-\frac{15}{4}$	$\frac{1}{8}$	1	$\frac{9}{8}$
Objective function row	0	$-\frac{25}{24}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$

Now, the negative element in the objective function row is $-\frac{25}{24}$. This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

- row : $\left(-\frac{1}{48}\right) \Big/ \left(-\frac{1}{24}\right) = \frac{1}{2}$

s_2 - row : $\left(\frac{9}{82}\right) \Big/ \left(-\frac{15}{4}\right) = -\frac{3}{10}$

Minimum of $\left\{\frac{1}{2}, -\frac{3}{10}\right\} = -\frac{3}{10}$

Hence select s_2 as the leaving variable. The pivotal element is $-\frac{15}{4}$. We get the following tableau at the end of Iteration No. 2.

	x	y	s_1	s_2	b – value
x - row	1	0	$-\frac{1}{480}$	$-\frac{1}{90}$	$-\frac{1}{30}$
y - row	0	1	$-\frac{1}{30}$	$-\frac{4}{15}$	$-\frac{3}{10}$
Objective function row	0	0	$-\frac{1}{18}$	$-\frac{5}{18}$	$-\frac{1}{3}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of $\frac{1}{V}$ is $-\frac{1}{3}$. This is provided by x = $-\frac{1}{30}$ and y = $-\frac{3}{10}$.

Thus the optimum value of the game is got as V = - 3. Using the relations

$\frac{r}{V} = x, \frac{s}{V} = y$, we obtain $r = \frac{1}{10}$ and $s = \frac{9}{10}$.

Problem 4

Transform the following game into an LPP

$$s = \frac{9}{10}$$

Solution

We have to determine the optimal strategy for player B. Using the entries of the given matrix, we obtain the inequalities

$$\begin{aligned} r + 8s + 3t &\le V, \\ 6r + 4s + 5t &\le V, \\ s + 2t &\le V \end{aligned}$$

Dividing by V, we get
$$\begin{aligned} \frac{r}{V} + 8\frac{s}{V} + 3\frac{t}{V} &\le 1, \\ 6\frac{r}{V} + 4\frac{s}{V} + 5\frac{t}{V} &\le 1, \\ \frac{s}{V} + 2\frac{t}{V} &\le 1 \end{aligned}$$

subject to the condition

$$r + s + t = V.$$

Consequently, we have $\frac{r}{V} + \frac{s}{V} + \frac{t}{V} = \frac{1}{V}$

.

Substitute

$$\frac{r}{V} = x, \quad \frac{s}{V} = y, \quad \frac{t}{V} = w.$$

Then we have the relations

$$x + 8y + 3w \le 1,$$
$$6x + 4y + 5w \le 1,$$
$$y + 2w \le 1.$$

We have to minimise V. i.e, We have to maximise $= \frac{r}{V} + \frac{s}{V} + \frac{t}{V}$. i.e, We have to maximise x + y + w.

Thus, the given game is transformed into the following equivalent LPP:

maximise x + y + w

subject to the restrictions

$$x + 8y + 3w \le 1,$$
$$6x + 4y + 5w \le 1,$$
$$y + 2w \le 1,$$
$$x \ge 0, y \ge 0, w \ge 0.$$

Questions

1. Explain how a game theory problem can be solved as an LPP.

2. Transform the game

$$\begin{array}{cc} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X_1 \\ X_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

into an LPP.

3. Using simplex method for LPP, solve the following game:

$$\begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$$

Answer: $r = \frac{1}{2}$, $s = \frac{1}{2}$, V = 4

4. Solve the following game with LPP approach:

$$\begin{bmatrix} 10 & 6 \\ 4 & 8 \end{bmatrix}$$

Answer: , , V = 7

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Lesson 9 - Goal Programming Formulation

Lesson Outline

- Introduction to goal programming
- Formulation of goal programming problems

Learning Objectives

After reading this lesson you should be able to

- Understand the importance of goal programming
- Formulate goal programming problems

Introduction

Generally speaking, the objective of a business organization is to maximize profits and to minimize expenditure, loss and wastage. However, a company may not always attempt at profit maximization only. At times, a necessity may arise to pay attention to other objectives also. We describe some such situations in the sequel.

A manufacturing organization may like to ensure uninterrupted supply of its products even if it means additional expenditure for the procurement of raw materials or personal delivery of goods during truckers' strike, etc. with the objective of assuring the good will of the customers.

A company may be interested in the full utilization of the capacity of the machines and therefore mechanics may be recruited for attending to break downs of the machines even though the occurrence of such break downs may be very rare.

A company, driven by social consciousness, may spend a portion of its profits on the maintenance of trees, parks, public roads, etc. to ensure the safety of the environment, with the objective earning the support of the society.

Another organization may have the objective of establishing brand name by providing high quality products to the consumers and for this purpose it may introduce rigorous measures of quality checks even though it may involve an increased expenditure.

While all the sales persons in a company are formally trained and highly experienced, the management may still pursue a policy to depute them for periodical training in reputed institutes so as to maximize their capability, without minding the extra expenditure incurred for their training.

A travel agency may be interested to ensure customer satisfaction of the highest order and as a consequence it may come forward to operate bus services even to remote places at the normal rates, so as to retain the customers in its fold.

A bank may offer services beyond normal working hours or on holidays even if it means payment of overtime to the staff, in order to adhere to the policy of customer satisfaction on priority basis.

A business organization may accord priority for the welfare of the employees and so a major part of the earnings may be apportioned on employee welfare measures.

A garment designer would like to be always known for the latest fashion and hence may spend more money on fashion design but sell the products at the normal rates, so as to earn the maximum reputation.

A newspaper may be interested in earning the unique distinction of 'Reporter of Remote Rural Areas' and so it may spend more money on journalists and advanced technology for communication.

The above typical instances go to show that the top level management of a business organization may embark upon different goals in addition to profit maximization. Such goals may be necessitated by external events or through internal discussion. At times, one such goal may be in conflict with another goal.

'Goal programming' seeks to deal with the process of decision making in a situation of multiple goals set forth by a business organization. A management may accord equal priority to different goals or sometimes a hierarchy of goals may be prescribed on their importance. One has to strive to achieve the goals in accordance with the priorities specified by the management. Sometimes the goals may be classified as **higher level goals and lower level goals** as perceived by the management and one would be interested in first achieving the higher order goals and afterwards considering lower order goals.

Some of the goals that may be preferred by a business organization are : maximum customer satisfaction, maximum good will of the customers, maximum utilization of the machine capacity, maximum reliability of the products, maximum support of the society, maximum utilization of the work force, maximum welfare of the employees, etc.

Since different goals of an organization are based on different units, the goal programming has a multi-dimensional objective function. This is in contrast with a linear programming problem in which the objective function is uni-dimensional.

Given a goal of an organization, one has to determine the conditions under which there will be under-achievement and over-achievement of the goal. The ideal situation will be the one with neither under-achievement nor over-achievement of the goal.

Formulation of Goal Programming Problems

In the sequel, we consider illustrative situations so as to explain the process of problem formulation in goal programming.

Notations

If there is a single goal, we have the following notations:

Let Du denote the under-achievement of the goal.

Let Do denote the over-achievement of the goal.

If there are two goals, we have the following notations:

Denote the under-achievement and the over-achievement of one goal by Du1 and Do1 respectively.

Denote the under-achievement and the over-achievement of another goal by Du2 and Do2 respectively.

Problem 1

Alpha company is known for the manufacture of tables and chairs. There is a profit of Rs. 200 per table and Rs. 80 per chair. Production of a table requires 5 hours of assembly and 3 hours in finishing. In order to produce a chair, the requirements are 3 hours of assembly and 2 hours of finishing. The company has 105 hours of assembly time and 65 hours of finishing. The company manager is interested to find out the optimal production of tables and chairs so as to have a maximum profit of Rs. 4000. Formulate a goal programming problem for this situation.

Solution

The manager is interested not only in the maximization of profit but he has also fixed a target of Rs. 4000 as profit. Thus, the problem involves a single goal of achieving the specified amount of profit.

Let D_u denote the under achievement of the target profit and let D_o be the over achievement.

The objective in the given situation is to minimize under achievement. Let Z be the objective function. Then the problem is the minimization of $Z = D_u$.

Formulation of the constraints

Let the number of tables to be produced be x and let the number of chairs to be produced be Y.

Profit from x tables = Rs. 200 x

Profit from y chairs = Rs. 80 y

The total profit = Profit from x tables and y chairs

+ under achievement of the profit target

-over achievement of the profit target

So we have the relationship 200 x + 80 y + D_u - D_o = 4000.

Assembly time

To produce x tables, the requirement of assembly time = 5 x hours. To produce y

chairs, the requirement is 3 y hours. So, the total requirement is $5x+3y$ hours. But the available time for assembly is 105 hours. Therefore constraint

$$5x + 3y \leq 105$$

must be fulfilled

Finishing time

To produce x tables, the requirement of finishing time = 3 x. To produce y chairs, the requirement is 2 y . So, the total requirement is 3 x+ 2 y. But the availability is 65 hours. Hence we have the restriction

$$3x + 2y \leq 65$$

Non-negativity restrictions

The number of tables and chairs produced, the under achievement of the profit target and the over achievement cannot be negative. Thus we have the restrictions

$$x \geq 0,\ y \geq 0,\ D_u \geq 0,\ D_o \geq 0$$

Statement of the problem

$$\text{Minimize } Z = D_u$$

subject to the constraints

Problem 2

Sweet Bakery Ltd. produces two recipes A and B. Both recipes are made of two food stuffs I and II. Production of one Kg of A requires 7 units of food stuff I and 4 units of food stuff II whereas for producing one Kg of B, 4 units of food stuff I and 3 units of food stuff II are required. The company has 145 units of food stuff I and 90 units of food stuff II. The profit per Kg of A is Rs. 120 while that of B is Rs. 90. The manager wants to earn a maximum profit of Rs. 2700 and to fulfil the demand of 12 Kgs of A. Formulate a goal programming problem for this situation.

Solution

The management has two goals.

To reach a profit of Rs. 2700

Production of 12 Kgs of recipe A.

Let D_{up} denote the under achievement of the profit target.

Let D_{op} denote the over achievement of the profit target.

Let D_{uA} denote the under achievement of the production target for recipe A.

Let D_{oA} denote the over achievement of the production target for recipe A.

The objective in this problem is to minimize the under achievement of the profit target and to minimize the under achievement of the production target for recipe A.

Let Z be the objective function. Then the problem is the minimization of

$$Z = D_{up} + D_{uA}$$

Constraints

Suppose the company has to produce x kgs of recipe A and y kgs of recipe B in order to achieve the two goals.

Condition on profit

Profit from x kgs of A = 120 x

Profit from y kgs of B = 90 y

The total profit = Profit from x kgs of A + Profit from y kgs of B

+ under achievement of the profit target

– over achievement of the profit target

$$= 120x + 90y + D_{up} - D_{op}$$

Thus we have the restriction

$$120x + 90y + D_{up} - D_{op} = 2700$$

Constraint for food stuff I:

$$7x + 4y \leq 145$$

Constraint for food stuff II:

$$4x + 3y \leq 90$$

Non-negativity restrictions

$$x,\ y,\ D_{up},\ D_{op},\ D_{uA},\ D_{oA} \geq 0$$

Condition on recipe A

The target production of A = optimal production of A

+ under achievement in production target of A

– over achievement of the production target of A.

Thus we have the condition

$$x + D_{uA} - D_{oA} = 12$$

Statement of the problem

$$\text{Minimize} \quad Z = D_{up} + D_{uA}$$

subject to the constraints

Questions

1. Explain the necessity of a goal programming.
2. Describe some instances of goal programming.
3. Explain the formulation of a goal programming problem.

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Lesson 10 - Queueing Theory

Lesson Outline

- Introduction
- Basic concepts of a Queueing System
- Numerical Examples of Waiting line Models

Learning Objectives

After reading this lesson you should be able to

- Understand the nature and scope of a Queueing System
- Waiting line Models and the solutions to the problems of Waiting line Models

Introduction

A business organization may be either production oriented or service oriented or both. With the rapid advancement of science and technology, the service sector has registered a phenomenal growth in the recent years. Customer satisfaction being the hallmark of an efficient service organization, a customer would like to receive the desired service as soon as he enters the organization or at least within a reasonable time of his arrival at the spot. **Queueing theory** is associated with the objective of improving the efficiency of service in any organization engaged in offering a service to the customers. If there are sufficient facilities in the service point to offer the desired service as soon as a customer enters, then there is no necessity to form a queue. Accumulation of customers at a service point leads to the formation of a queue.

The flow of customers from a finite or infinite population towards the service point leads to the formation of a ***queue (waiting line)***. When there is exact match between the service facilities and the arrival of customers, ***waiting time*** is caused either for the service facilities or for the arrival of the customers. A queueing system consists of one or more queues and one or more servers. ***A queueing system*** works with a class of procedures. The incoming customer either waits at the queue sometimes or gets his turn to be served based on the working conditions prevalent in the queueing system. If the server is free at the time of arrival of a customer, then the customer can directly enter into the counter for getting his service. Thus there are two possible things: the system may experience "customer waiting" and /or "idle time of the server".

System Approach to a Queue

A system consists of three components, namely, (i) input, (ii) processes and (iii) output. As regards a queueing system, the input is constituted by the customers who arrive at a service point in anticipation of a service. The process includes the methods of the service offered by the organization, the behavior of the customers and the discipline of the queue. Hence, the following data are required to understand and analyze a queueing system:

i. The input (the pattern of arrival of the customers to the service point)
ii. The service mechanism (the pattern of service in the organization)
iii. The queue discipline (the principle under which the queue operates) and
iv. The behaviour of the customers

The input (the pattern of arrival)

One has to find out the way in which the customers arrive at a service point and join the queue. Customers normally arrive almost in a random way. It is highly difficult to guess the pattern of arrival of the customers. Therefore we have to associate probabilities with the arrival of the customers and hence the probability distribution for ***inter-arrival*** times (the time between two successive arrivals of the customers) has to be found out. We take up a queueing system with the assumption that the customers arrive in a Poisson process. We also assume that the mean arrival rate of the customers is found it to be δ.

The Service Mechanism

The term service mechanism refers to the arrangement of service facility to serve the customers. If there is infinite number of servers, then all the customers are served instantaneously as soon as they arrive and a queue will not be found. If the system consists of a finite number of servers, then the customers are served according to a pre-determined rule by making the server service time a constant or a random variable. Distribution of service time follows '***Exponential distribution***'.

Queueing Discipline

The term queueing discipline refers to a procedure by which the customers are selected from the queue for offering the service. The following disciplines are generally adopted by a queueing system:

First Come First Served – (FCFS)
First In First Out – (FIFO)
Last In First Out – (LIFO)
Selection for service In Random Order (SIRO)

The Behaviour of the Customers

In general, one may observe the following modes of behavior of the customers in a queue:

Normally, the customers arrive one by one into the system. However, there is a possibility for another phenomenon. The term ***Bulk arrival*** refers to the arrival of customers in groups.

Consider the case of several service counters in the organization. Then each service counter will have a queue. When there are several queues, the customers from one queue may switch over to another queue if it is of smaller size. Such a behaviour of the customers is referred to as ***Jockeying.***

Sometimes a customer on arrival may not join a queue after observing that the queue length is very large. This behavior of the customers is called ***Balking*** of the customers.

In certain cases, a customer already present in a queue may leave the queue thinking that the waiting time may be too much. This behaviour of the customers is called ***Reneging.***

Notations

Customers in the system refers to the customers receiving service at the service point and the customers who are waiting to receive the service.

The following notations are used in Queueing theory:

n - No of customers in the system

C - No of servers in the system

$P_n(t)$ – The probability of having n customers in the system at time t

P_n - The steady state probability of having n customers in the system

P_0 - The probability of having zero customer in the system

L_q - Average number of customers waiting in the queue

L_s - Average number of customers waiting in the system (in the queue and in the service points)

W_q - Average waiting time of the customers in the queue

W_s - Average waiting time of the customers in the system (in the queue and in the service points)

δ - Arrival rate of the customers

μ - Service rate of the server

ϕ - Utilization factor of the server

δ eff - Effective rate of the arrival of customers

M - Poisson distribution

N - Maximum number of customers allowed in the system. It also refers to the size of the calling source of the customers

GD - General discipline for service in the organization like first in first –

served (FIFS), last-in-first served (LIFS), etc.

Traffic Intensity (Or Utilization Factor)

The traffic intensity of a queue is denoted by ϕ. It is defined by the rule

ϕ = (Mean arrival time)/(Mean service time) $= \delta/\mu \ (< 1)$

and the unit of traffic intensity is called Erlang.

Types of Waiting Line Models

In general, a queueing model can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are as follows.

P- Arrival rate distribution i.e probability law for the arrival /inter – arrival time.

Q - Service rate distribution, i.e probability law according to which the customers are being

served

R - Number of Servers (i.e., number of service points)

X - Service discipline

Y - Maximum number of customers allowed in the system

Z - Size of the calling source of the customers

A queuing model with the above parameters is denoted by the symbol (P/Q/R : X/Y/Z)

Model 1 : (M/M/1) : (GD/ ∞/ ∞) Model

This is a waiting line model with the following properties:

1. The arrival rate follows Poisson (M) distribution
2. The service rate follows Poisson distribution (M)
3. The number of servers in the system is 1
4. The service discipline is general discipline (i.e., GD)
5. Maximum number of customers allowed in the system is infinite (**∞**)
6. he size of the calling source is infinite (**∞)**

The steady state equations to obtain Pn, the probability of having n customers in the system, and the formulae for obtaining the values for Ls, Lq, Ws and Wq are as follows:

$$n = 0,1,2,\dots \infty \text{ where } \phi = \delta/\mu < 1$$

Ls – Average number of customers waiting in the system (i.e., waiting in the queue and the service station)

$$Pn = \phi^n (1-\phi)$$

$$Ls = \phi/(1- \text{"}\phi\text{"})$$

$$Lq = Ls - \delta/\mu -$$

$$= \phi/(1- \text{"}\phi\text{"}) - \phi$$

$$= (\text{"} \phi\text{"} - (1-\text{"}\phi\text{"}) \text{"}\phi\text{"})/(1-\text{"}\phi\text{"})$$

$$= \phi^2/(1-\text{"}\phi\text{"})$$

Average waiting time of customers in the system (in the queue and in the service point) = Ws

$$= Ls/\delta = \phi/((1 - \text{"}\phi\text{"})\text{"}\delta\text{"})$$

$$= \phi/((1 - \text{"}\phi\text{"})) \text{ x } 1/\mu\phi \quad (\text{since } \delta = \mu\phi)$$

$$= 1/(\text{"}\mu\text{"} -\text{"}\mu\phi\text{"})$$

$$= 1/(\text{"}\mu\text{"} -\text{"}\delta\text{"})$$

Wq = Average waiting time of customers in the queue

$$= Lq/\delta = 1/\delta \text{ x } \phi^2/(1-\text{"}\phi\text{"})$$

$$= 1/\mu\phi \text{ x } \phi^2/(1-\text{"}\phi\text{"}) \quad (\text{since } \delta = \mu\phi)$$

$$= \phi/(\text{"}\mu\text{"} -\text{"}\mu\phi\text{"})$$

$$= \phi/(\text{"}\mu\text{"} -\text{"}\delta\text{"})$$

Summary of Formulae

1. $\phi = \delta/\mu$
2. $Pn = \phi^n (1-\phi)$
3. $P0 = 1 - \phi$
4. $Ls = \phi/(1 - \text{“}\phi\text{”})$
5. $Lq = \phi^2/(1 - \text{“}\phi\text{”})$
6. $Ws = 1/(\text{“}\mu\text{”} - \text{”}\delta\text{”})$
7. $Wq = \phi/(\text{“}\mu\text{”} - \text{”}\delta\text{”})$

Example 1

The arrival rate of customers at a petrol bunk follows a Poisson distribution with a mean of 27 per hour. The petrol bunk has only one unit of service. The service rate at the petrol bunk also follows Poisson distribution with mean of 36 per hour. Determine the following:

What is the probability of having zero customer in the system ?

What is the probability of having 6 customers in the system ?

What is the probability of having 10 customers in the system ?

The values of Ls, Lq, Ws and Wq

Solution

Given that the arrival rate follows Poisson distribution with mean = 27

With our notations, $\delta = 27$ per hour

Given that the service rate follows Poisson distribution with mean = 36

Hence $\mu = 36$ per hour

Consequently, the utilization factor $\phi = \delta/\mu = 27/36 = 3/4 = 0.75$

The probability of having zero customer in the system is

$$P0 = \phi^0 (1 - \phi)$$
$$= 1 - \phi$$
$$= 1 - 0.75$$
$$= 0.25$$

The probability of having 6 customers in the system is

$$P6 \quad = \phi^6 (1- \phi)$$
$$= (0.75)6\ (1- 0.75)$$
$$= 0.0178 \times 0.25$$
$$= 0.0445$$

The probability of having 10 customers in the system is

$$P10 \quad = \phi 10\ (1- \phi)$$
$$= (0.75)10\ (1- 0.75)$$
$$= 0.0563 \times 0.25 \quad = 0.0141$$

Ls = ϕ/(1- "ϕ")= 0.75/(1- 0.75) = 0.75/0.25 = **3 customers**

Lq = ϕ^2/(1-"ϕ") = (0.75)^2/(1-0.75) = 0.5625/0.25 =**2.25 customers**

Ws = 1/("μ" -"δ") = 1/(36 -27) = 1/9 =**0.011 hour**

Wq = ϕ/("μ" -"δ") = 0.75/(36-27) = 0.75/(9)= **0.0833 hour**

Note: In the determination of Ls and ¬ Lq, we are not rounding the values to the nearest integer.

Example 2

At one-man book binding centre, customers arrive according to Poisson distribution with mean arrival rate of 4 per hour and the book binding time is exponentially distributed with an average of 12 minutes. Find out the following:

The average number of customers in the book binding centre and the average number of customers waiting for book binding.

The percentage of time arrival can walk in straight without having to wait.

The percentage of customers who have to wait before getting into the book binder's table

Solution

Given mean arrival of customers $\delta = 4/60 = 1/15$

Mean time for server $\mu = 1/12$

Hence $\phi = \delta / \mu = [1/15] \times 12 = 12/15 = 0.8$

The average number of customers in the system (Customers in the queue and in the book binding centre)

$$Ls = \phi / 1 - \phi = 0.8 / 1 - 0.8$$

$$= 0.8 / 0.2$$

$$= 4$$

$$= 4 \text{ Customers}$$

The percentage of time arrival can walk straight into book binder's table without waiting is

service utilization $= \phi\ \%$

$= \delta / \mu \%$

$= 0.8 \times 100$

$= 80$

The percentage of customers who have to wait before getting into the book binder's table $= (1-\phi)\%$

$(1-0.8)\% \quad = 0.2 \times 100$

$= 20\%$

Example 3

A person repairing wrist watches observes that the time spent on the wrist watches has an exponential distribution with mean 20 minutes. If the wrist watches are repaired in the order in which they come in and their arrival is Poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day? On an average, how many jobs are ahead of a wrist watch just brought in?

Solution

The arrival rate $\delta = 15/8x60 = 1/32$ units/minute

The service rate $\mu = 1/20$ units/minute

$$\phi = \delta/\mu$$

$= 1/32 \ / \ 1/20$
$= 20/32 = 5/8$

The number of jobs ahead of the wrist watch just brought in = the average number of jobs in the system

$$
\begin{aligned}
Ls &= \phi / 1-\phi \\
&= (5/8) / \{1-(5/8)\} \\
&= (5/8) / (3/8) \\
&= 5/3.
\end{aligned}
$$

The number of hours for which the repairman remains busy in an 8-hour day = 8 x δ/μ = 8x (5/8) =5 hours.

Hence the time for which repairman remains idle in an 8-hour day = 8-5= 3 hours.

Example 4

Customers are arriving at a service centre at the rate of 70 per hour. The average service time for a customer is 45 seconds. The arrival rate and service rate follow Poisson distribution. There is a complaint that the customers wait for a long duration. The proprietor of the centre is ready to consider the installation of one more service point to reduce the average time to 35 seconds if the idle time of the service point is less than 9% and the average queue length at the service point is more than 8 customers. Determine whether the installation of the second service point is worth from the point of view of the proprietor.

Solution

Arrival rate of the customers at the service centre δ =70 per hour
Average service time for a customer = 45 Seconds

$$
\begin{aligned}
\text{Service rate} \quad \mu &= \frac{1 \text{ hour}}{45 \text{ seconds}} \\
&= 3600/45 \\
&= 80 \\
&= 80 \text{ customers per hour}
\end{aligned}
$$

Utilization factor $\phi = \delta/\mu$

$= 70/80$

$= 0.875$

(a) Waiting no. of customers in the queue is Lq

$$Lq = \phi^2/((1 - \text{“}\phi\text{”})) = \frac{(0.875)2}{1-0.875}$$

$$= \frac{0.7656}{0.125}$$

$$= 6.125$$

$$= 6 \text{ Vehicles}$$

(b) Desired service time for a customer (after addition another service point) =35 seconds

The new service rate after installation of an additional service point = 1 hour/35 Seconds= 3600/35

=102.68 Customers / hour

Utilization factor $\phi = \delta/\mu = 70 / 102.86$

$= 0.681$

Percentage of idle time of the service point = $(1-\phi)\%$

$= (1-0.681)\%$

$= 0.319\%$

$= 31.9$

$= 32\%$

This idle time is not less than 9% which is desired by the proprietor.

Hence, the installation of the second service point is not justified since the average waiting number of customers in the queue is more than 8 but the idle time is not less than 32%.

Model 2 : (M/M/C) : (GD/ ∞/ ∞) Model

The following assumptions are made for this model:

The arrival rate follows Poisson distribution

The service rate follows Poisson distribution

The number of servers is C

The service discipline is general discipline

The maximum number of customers allowed in the system is infinite

With these assumptions, the steady state equation for the probability of having n customers in the system is given by

$$P_n = \frac{\phi^n P_o}{n!} \quad ,o \leq n \leq C$$

$$= \frac{\phi n \ \ Po}{C\ n\text{-}C\ C!} \quad \text{for } n > C \text{ where } \phi / C < 1$$

where $[\delta / \mu C] < 1$ as $\phi = \delta / \mu$

$$P0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi C \ / (C! \ [1 - \phi/C])\}\text{-}1$$

where C! = 1 x 2 x 3 x …………….. upto C

$$Lq = [\phi\ C+1 / [C\text{-}1! \ (C - \phi)] \ ] \times P0$$

$$= (C\phi\ PC) \ / \ (C - \phi)2$$

$$Ls = Lq + \phi \quad \text{and} \quad Ws = Wq + 1 / \mu$$

$$Wq = Lq / \delta$$

Particular cases

Under special conditions $Po = 1 - \phi$ and $Lq = \phi^{C+!} / C\ 2$ where $\phi < 1$ and

$Po = (C\text{-}\phi)\ (C - 1)! / C\ C$

and $Lq = \phi / (C\text{-}\phi)$, where $\phi / C < 1$

Example 1

At a Toll Gate, vehicles arrive at the rate of 24 per hour and the arrival rate follows Poisson distribution. The time to collect a toll and permitting the vehicle to pass follows exponential distribution and the passing rate is 18 vehicles per hour. There are 4 passing counters. Determine the following:

1. Po and P3
2. Lq, Ls, Wq and Ws

Solution

The arrival rate δ= 24 per hour.

The passing rate μ = 18 Per hour.

No. of passing counters C=4.

$\phi = \delta / \mu$

$= 24 / 18$

$=1.33$

(i) $P0 = \{[\sum_{n=0}^{C-1} \phi n/n!] + \phi C / (C! [1 - \phi/C])\}-1$

$= \{[\sum_{n=0}^{3} (1.33)n/n!] + (1.33)4 /(4! [1 - (1.33)/ 4])\}-1$

$= \{ (1.33)0 / = 0! + (1.33)1 / 1! + (1.33)2 / 2! + (1.33)3 / 3! + (1.33)4 / 24! [1 - (1.33)/ 4] \}-1$

$=[1 + 1.33 + 0.88 + 0.39 + 3.129/16.62] -1$

$=[3.60 + 0.19]-1$

$=[3.79]-1$

$= 0.264$

We know Pn $= (\phi n / n!) Po$ for $0 \leq n \leq C$

P3 $= (\phi 3 / 3!) Po$ since $0 \leq 3 \leq 4$

$= (1.33)3 / 6] x 0.264$

$= 2.353 x 0.044$

$= 0.1035$

(ii) Lq $= \frac{\phi^{C+1} XP0}{(C - 1)! (C-\phi)2}$

$$= \frac{(1.33)5 \text{ X } 0.264}{3! \text{ X } (4 - 1.33)2}$$

$$= \frac{(4.1616) \text{ X } 0.264}{6 \text{ X } (2.77)2}$$

$$= \frac{(4.1616) \text{ X } 0.264}{46 .0374}$$

= 1.099 / 46.0374

= 0.0239

= 0.0239 Vehicles

Ls = Lq + ϕ = 0.0239 + 1.33

= 1.3539 Vehicles

Wq = Lq / δ = 0.0239 /24

= 0.000996 hrs

Ws = Wq + 1 /μ = 0.000996 + 1/18

= 0.000996 + 0.055555

= 0.056551 hours.

Example 2

In a bank, there are two cashiers in the cash counters. The service time for each customer is exponential with mean 4 minutes and the arrival rate of the customers is 10 per hour and the arrival of the customers follows Poisson distribution. Determine the following:

1. The probability of having to wait for service
2. The expected percentage of idle time for each cashier
3. Whenever a customer has to wait, how much time he is expected to wait in the Bank?

Solution

$$P0 = \{[\sum_{n=0}^{C-1} \phi n/n!] + \phi C \ / \ (C! \ [1 - \phi/C])\}-1$$

Where ϕ =δ / μ given arrival rate = 10 per hour

= 10 / 60 = 1 / 6 per minute

Service rate = 4 minutes

$\mu = 1 / 4$ person per minute

Hence $\phi = \delta / \mu = (1 / 6) \times 4 = 2 / 3$

$= 0.67$

$$P0 = \{[\sum_{n=0}^{1} \phi n/n!] + (0.67)2 \ / \ (2! \ [1 - 0.67/2])\}\text{-}1$$

$= [1 + (\phi / 1!) \ ] + 0.4489 / (2 - 0.67)]\text{-}1$

$= [1 + 0.67 + 0.4489 / (1.33)]\text{-}1$

$= [1 + 0.67 + 0.34]\text{-}1$

$= [2.01]\text{-}1$

$= 1 / 2$

i. The Probability of having to wait for the service is P (w > 0)

$$= \frac{\phi \, C \quad X \, P0}{C! \ [1 - \phi/C]}$$

$$= \frac{0.67 \; 2X \; (1 / 2)}{2! \ [1 - 0.67 /2]}$$

$= 0.4489 / 2.66$

$= 0.168$

ii. The probability of idle time for each cashier is

$= 1\text{-} P (w > 0)$

$= 1\text{-}1/3$

$= 2/3$

Percentage of time the service remains idle = 67% approximately

iii. The expected length of waiting time (w/w>0)

$= 1 / (C \mu - \delta)$

$= 1 / \ [(1 / 2) - (1 / 6) \ ]$

$= 3$ minutes

Questions

1. Explain the basic concepts of Queuing theory.

2. The arrival rate of customers at a booking counter in a railway station follows Poisson distribution with a mean of 45 per hour. The service rate of the booking clerk also follows Poisson distribution with a mean of 60 per hour.

i. What is the probability of having 0 customer in the system?

ii. What is the probability of having 5 customers in the system?

iii. What is the probability of having 10 customers in the system?

iv. Determine Ls, Lq,Ws and Wq

Answer

i. 0.25

ii.0.0593

iii.0.0141

iv. Ls=3 customers, Lq=2.25 customers, Ws =0.067 hour and Wq = 0.05 hour.

3. The arrivals and services in a service centre follow Poisson Distribution. The arrival rate of the customers is 8 per hour. The service rate is 10 customers per hour. Find out the following:

i. The average number of the customers waiting for service.

ii. The average time a customer has to wait in the queue.

iii. The average time a customer has to be in the system.

Answer

i. 3.2 customers

ii. 0.4 hours

iii.0.3 hours